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DEFINITIONS OF THE DISCRIMINANT OF A RATIONAL INTEGRAL FUNCTION OF ONE VARIABLE.

By G. A. MILLER, University of Illinois.

Many students meet the term *discriminant* for the first time in the discussion of the solutions of the quadratic equation $ax^2 + bx + c = 0$. It is here commonly defined as the expression $b^2 - 4ac$ and it is usually noted that the vanishing of this expression is a necessary and sufficient condition that the given quadratic equation has equal roots. As the vanishing of this expression implies the vanishing of many other expressions and vice versa, it is clear that the given expression is by no means the only one whose vanishing is a necessary and sufficient condition that the quadratic equation under consideration has equal roots. Most of these other expressions might be said to be less simple but they include $4ac - b^2$ which appears to be just as simple as the expression defined above as the discriminant.

We emphasize these evident facts because the mathematical student is apt to meet in his more advanced work definitions of the term discriminant which are either vague or contradictory with his earliest use of the term. As an instance of the latter type of definitions we may cite the one given on page 96 of the *Theory of Equations* by F. Cajori, 1912. The discriminant of $f(x)$ is here defined as the resultant of $f(x)$ and $f'(x)$. If we find this resultant, when

$$f(x) = ax^2 + bx + c,$$

in accord with the illustration of the term resultant found on the same page, we obtain the following determinant:

$$\begin{vmatrix} a & b & c \\ 0 & 2a & b \\ 2a & b & 0 \end{vmatrix} = ab^2 - 4a^2c.$$

which is a times the discriminant of this quadratic as defined in the preceding paragraph.

In order to agree with the special case under consideration the discriminant of $f(x)$ might therefore be defined as the resultant of $f(x)$ and $f'(x)$ divided by the coefficient of the highest power of x in the rational integral function of x denoted by $f(x)$, provided we would adopt the definition of resultant implied in the illustration noted above. This definition, as well as the one given on page 196 of Bôcher's *Introduction to Higher Algebra*, 1907, is, however, not in accord with that found in some other standard works. In particular, these definitions do not coincide with the one found in the *Encyclopédie des Sciences Mathématiques*, tome 1, volume 2, page 75, but the resultants obtained according to these various definitions may differ only with respect to sign. This difference actually presents itself in the special case considered in the preceding paragraph, as can easily be verified.

We thus see that the reader may be considerably perplexed when he tries to harmonize the definitions of the term discriminant found in more advanced works with the earliest example of a discriminant with which he became acquainted, unless he is forewarned that the current definitions relating to this term are not in perfect agreement. The definition given on page 250 of Bôcher's *Introduction to Higher Algebra*, 1907, does not clear up the matter since it applies only to the case when the coefficient of the highest power of the unknown is unity. It may perhaps be taken for granted by many readers that the definition found in the large mathematical encyclopedias[1] will eventually be universally adopted. If this will be done we shall have to say that the discriminant of $ax^2 + bx + c$ is $4ac - b^2$ instead of $b^2 - 4ac$.

The question arises whether this change is desirable in view of the fact that $b^2 - 4ac$ appears under the radical sign in the common solution of the equation $ax^2 + bx + c = 0$ and hence it seems convenient to call it the discriminant of the quadratic equation rather than to say that this expression multiplied by -1 is the discriminant of the given equation. On the other hand, uniformity in definitions is so desirable that this slight convenience might well be sacrificed for the sake of avoiding the confusion caused by contradictory definitions of mathematical terms.

Another solution of this difficulty would be secured if mathematicians could agree on not following the large mathematical encyclopedias in this particular. In fact, such a well-known authority as the second edition of Pascal's *Repertorium der höheren Mathematik* has thrown its weight in favor of this movement by adopting a definition according to which the discriminant of $f(x)$ is the resultant of $f(x)$ and $f'(x)$ multiplied by

$$\frac{1}{a_0}(-1)^{n(n-1)/2},$$

[1] *Encyklopädie der Mathematischen Wissenschaften*, Vol. 1, p. 251; *Encyclopédie des Sciences Mathématiques*, tome 1, Vol. 2, pp. 97, 100.

where a_0 is the coefficient of the highest power of x in $f(x)$ and n is the degree of this function.[1] If this definition and the usual definition of resultant are adopted the discriminant of $ax^2 + bx + c$ is $b^2 - 4ac$, which agrees with common usage in our textbooks.

When J. J. Sylvester first used the term discriminant as a technical mathematical term, *Philosophical Magazine*, 1851, page 406, he added the following explanatory remark: "Discriminant because it affords the discrimen or test for ascertaining whether or not equal factors enter into a function of two variables, or more generally, of the existence or otherwise of multiple points in the locus represented or characterized by an algebraical function." Several authors have used somewhat similar explanatory words as a definition. Thus we find in the *Theory of Equations* by F. Cajori, 1912, page 96, "The discriminant of an equation $f(x) = 0$ may be otherwise defined as the simplest function of the coefficients, or of the roots, whose vanishing signifies that the equation has equal roots." Such a definition is clearly too vague to determine a definite expression as the discriminant of a given function. This is also true of such definitions as the following: "The discriminant of an equation involving a single unknown is the simplest function of the coefficients in a rational integral form, whose vanishing expresses the condition for equal roots," *Theory of Equations* by Burnside and Panton, volume 2, 1904, page 83; cf. *Theory of Equations* by F. Cajori, 1912, page 97.

These observations may suffice to forewarn the young mathematician in regard to the different meanings assigned to common mathematical terms by various authors. Although the term discriminant is sixty-seven years old and has found a place in many elementary textbooks it is still being defined differently by eminent authorities, so that it would even now be difficult to determine by the process of comparing authorities what expression should be called the discriminant of the general quadratic equation in one unknown. In addition to the noted difficulties in the way of such a determination we may add that on page 315 of the first volume of Borel's *Die Elemente der Mathematik* translated by P. Stäckel, 1908, it is stated that

$$\frac{b^2 - 4ac}{4}$$

is sometimes called the discriminant of the equation $ax^2 + bx + c = 0$.

It is a very singular fact that the definition of the discriminant of $f(x)$ contained in what are now universally regarded as our foremost works of reference, viz., the large mathematical encyclopedias in course of publication, is not in accord with the common textbook use of this term in the special case when $f(x) = ax^2 + bx + c$. The main objects of this note are to direct more general attention to this fact and to save the beginner from the perplexities into which

[1] Pascal's *Repertorium der höheren Mathematik*, Vol. 1 (1910), p. 274. In the 1900 edition of this work, p. 88, the definition of the discriminant of $f(x)$ agrees with that found in the large mathematical encyclopedias.

he is naturally led by definitions found in some of our reliable American textbooks. In a previous note published in this MONTHLY, volume 24, 1917, page 106, we directed attention to desirable changes in the definitions of the term discriminant found in two of our standard dictionaries and in the *International Encyclopedia*.

In view of the fact that such eminent authorities give different definitions of the term discriminant of $f(x)$ it may appear almost presumptuous on the part of the present writer to express an opinion in regard to what appears to him as the most desirable definition to adopt. The advantages of uniformity along this line impel him, however, to say that he would adopt the definition found in the second edition of Pascal's *Repertorium* noted above, and hence he would not follow the large mathematical encyclopedias in this particular case. He would, on the other hand, follow these encyclopedias as regards the definition of the resultant of two functions of a single variable, and hence he would digress also in this instance from the definition adopted in Bôcher's *Algebra* cited above. These conclusions are based mainly on what appears to the writer as the most feasible steps towards securing uniformity.

Dr. A. J. Kempner recently directed my attention to the following definition: "By the *resultant* of two equations $f(x) = 0$ and $\phi(x) = 0$ is meant that integral function of the coefficients of $f(x)$ and $\phi(x)$ whose vanishing is the necessary and sufficient condition that $f(x) = 0$ and $\phi(x) = 0$ have a common root"; *College Algebra* by H. B. Fine, 1904, page 512. The fact that this definition appears in one of our most advanced American algebras may justify a reference to it here, especially since the definitions of the terms discriminant and resultant are so closely related. It is difficult to see how such a vague definition can fail to embarrass the serious student.

In this connection it may be of interest to refer to another term (*real curve*) which is used in college mathematics with a meaning differing from that assigned to it in some of our best reference books. To illustrate this fact we may note that in the *Encyclopédie des Sciences Mathématiques*, tome 3, volume 3, page 261, it is stated that "a real curve need not contain any real point; as happens, for instance, in the case of the real conic $x^2 + y^2 + 1 = 0$." On the contrary, it is customary to call such a conic in our textbooks an *imaginary* curve notwithstanding the fact that all the coefficients of its equation are real, and this usage seems to be also in accord with the terminology employed on page 71 of the volume just mentioned. The use of the term *real curve* for an imaginary ellipse in analytic geometry does not appear desirable, and hence it is the more questionable whether this use should be sanctioned in more advanced subjects.

WHAT IS THE ORIGIN OF THE NAME "ROLLE'S CURVE"?

By FLORIAN CAJORI, University of California.

It is a curious circumstance that the two topics of mathematical interest for which Michael Rolle is now best known—"Rolle's theorem" and "Rolle's curve" —are the ones which modern readers have been experiencing difficulty in finding in Rolle's published books or articles and in legitimately associating with his name. As late as 1903 A. v. Braunmühl[1] declared that "Rolle's theorem"—the theorem according to which $f'(x) = 0$ has at least one real root lying between two successive real roots of $f(x) = 0$—had not been found in Rolle's works and was therefore wrongly attributed to him. The present writer[2] was finally able to show that the theorem is rightly attributed to Rolle, it being found in an out-of-the-way publication of Rolle bearing the title *Démonstration d'une Methode pour résoudre les Egalitez de tous les degrez*, Paris, 1691.

As regards "Rolle's curve," historians have been thus far less fortunate. Some years ago Gino Loria[3] frankly admitted that he had not been able to ascertain the reason why the cubic curve $xy^2 = a(y + x)^2$ or $xy^2 = a(y - mx)^2$ should be named after Rolle. F. G. Teixeira[4] speaks of "Rolle's curve" and fully describes its properties, but gives no information as to Rolle's connection with the curve. Nor does Felix Müller,[5] who simply mentions its name. The earliest appearance that we have seen of this name in print is 1896, when Elgé[6] wrote an article "Sur la courbe de Rolle. Sa construction par points et par tangents."

In 1898 Gino Loria[7] published a note of inquiry regarding the origin of the name, and said that de Longchamps believed he had seen the name earlier in the *Educational Times*. As this note elicited no response, it was republished ten years later[8] with an annotation by H. Brocard who states that the discussion carried on between Saurin and Rolle on the "problème général des tangentes" in the *Journal des Savans* in the years 1702 and 1703 seemed to be a promising place for search, but that it is not given there; Brocard admits that he was not able to find any justification for attributing the curve to Rolle. No greater success rewarded the efforts of the present writer who examined the booklet quoted above, which contains "Rolle's theorem," also Rolle's *Traité d'Algebre*, Paris, 1690, and the different volumes of the *Histoire de l'Académie royale des*

[1] A. v. Braunmühl in *Bibliotheca Mathematica*, 3. F., Vol. 4, p. 399.

[2] F. Cajori in *Bibliotheca Mathematica*, 3. F., Vol. 11, 1910–1911, pp. 300–313.

[3] G. Loria, *Spezielle algebraische und transcendente Ebene Kurven*. Deutsch v. F. Schütte, Leipzig, 1902, p. 75; 2d ed., 1910, p.78.

[4] F. G. Teixeira, *Traité des courbes spéciales remarquables planes et gauches*, in *Obras*, Vol. 4, Coimbra, 1908, pp. 115, 134.

[5] F. Müller, *Mathematisches Vokabularium*, Leipzig, 1900, "Curve." See also F. Müller in *Bibliotheca Mathematica*, 3. F., Vol. 2, 1901, p. 302.

[6] Elgé in *Journal de mathématiques spéciales*, 4 S., Vol. 5, 1896, pp. 32–34. See also p. 55.

[7] *L'Intermédiaire des mathématiciens*, Vol. 5, 1898, p. 76. No. 1257.

[8] *L'Intermédiaire des mathématiciens*, Vol. 16, 1908, p. 244.

sciences at Paris. The curve and its graph are given without name in Cramer's *Introduction à l'analyse des lignes courbes*, 1750, pp. 188, 478, and in O. Gherli's *Elementi teorico-pratici delle matematiche pure*, Vol. IV, Modena, 1773, pp. 343, 345.

The question still remains to be solved, who first used the name "Rolle's curve" for $xy^2 = a(y + x)^2$ and what connection, if any, Rolle had with it.[1] Who can throw further light on this?

THE MATHEMATICS OF AËRODYNAMICS.[2]

By EDWIN BIDWELL WILSON, Massachusetts Institute of Technology.

1. In the development of a branch of engineering there are several stages, sometimes long drawn out and poorly differentiated, sometimes brief and sharply defined.

There is the stage in which the bold inventors, trying one way and another, often with personal danger or even disaster, accomplish the first crude beginnings. In ship construction this epoch is prehistoric; in steam engineering it starts, at least in story, with Watts; in electrical engineering it lies within the memory of men still living; in aëronautics it stretches over a long period admirably described by A. F. Zahm in his book *Aërial Navigation*. The first essays in aëronautics, if we overlook the legendary Icarus, were the balloon and the balloon-type machine; progress with the heavier-than-air type may well be dated from Langley and the Wrights, though experiments with gliders are known for some time previous to them.

There is another stage—that in which the properly equipped mathematician and physicist construct a theory to correspond to the branch of engineering and by the indications of that theory aid materially in hastening the development of improved types of machines, or at least, if they arrive too late to help in the major development, they assist in codifying the fundamental scientific principles which underlie the subject and make possible systematic training of the youth to carry on the advance in details. The development of mathematics and physics was sufficient before the advent of electrical engineering to enable major advances to be made by William Thomson (later Lord Kelvin), by Oliver Heaviside, and by M. I. Pupin, to mention only three. In the case of the simpler parts of mechanical engineering and of naval architecture, the pressing necessities of prehistoric and early historic man were such that the art far outstripped the

[1] As far as I now know, the only curve which prior to 1896 may be said to have been associated with the name of Rolle is mentioned by J. Bernoulli in a letter to Leibniz, dated May 7, 1701. Bernoulli there refers to the "ultima Rolii curva a Varignonio delineata

$$y = 2 + \sqrt{(4 + 2x)} + \sqrt{(4x)},"$$

Got. Gul. Leibnitii et Johan. Bernoullii Commercium Philosophicum et Mathematicum, Tomus secundus, Lausannæ & Genevæ, MDCCXLV, pp. 41, 42, 44.

[2] Address read before the Mathematical Association of America at its annual meeting, at Columbia University, New York, Saturday, December 30, 1916.

science; indeed the limitations of our mathematical and physical knowledge of the behavior of liquids are still so serious that when we wish a cup-defender we have to rely almost entirely upon the fine sense of a genius like Herreshoff instead of upon the theories and calculations of engineers like Professor C. H. Peabody and Rear-Admiral Taylor.

In the development of the airplane, however, since the early experimental and inventive successes of the Wrights and their competitors and followers in France, much has been accomplished in a single decade by the mathematician and physicist, in particular by G. H. Bryan, whose book *Stability in Aviation* was epochal, and by L. Bairstow, of the National Physical Laboratory, Teddington, England, whose experimental and theoretical work has done so much toward rendering Bryan's ideas directly and surely applicable to actual flying machines. I have been fortunate enough under commission from our government to add my mite to the theoretical advance (*Motion of an Airplane in Gusts*), and we in this country should all do much homage to the pioneer work which Lieut. J. C. Hunsaker, U. S. N., performed at the Massachusetts Institute of Technology.

There is a third stage, a relatively late stage in this uprearing of a branch of engineering. Here we find the scientific and technical knowledge so codified in text-books and manuals that it is not particularly hard for a school of technology, properly equipped with laboratories and teaching staff, to take in as raw material average young fellows and turn out as manufactured product tolerably competent engineers, able to design and make minor improvements in standard machines. This stage is not yet quite attained in aëronautics, but with the obviously great technical advances which have come about with such perplexing rapidity during the past two or three years under the pressure of war necessities in Europe and in this country, and with the presumably great theoretical advances which have accompanied them, it certainly will not take long after the war shall have terminated and after the present belligerent governments shall have ventured to divulge the ways and means and ends in aëronautics which they now guard so sedulously—it certainly will not then take long to work down the science and art of aëronautical engineering to the point where it takes its place in this third stage with mechanical, civil, and electrical engineering. Already a preliminary codification may be found in the course prepared by Messrs. Klemin and Huff, on the basis of the course given by their former teacher, Dr. Hunsaker, which is now appearing serially in the recently established, interesting and important journal *Aviation and Aëronautical Engineering.*

My subject as announced by your program committee is the "Mathematics of Aërodynamics," and if I have been long in my preamble it is merely because I feel that before we examine one special phase of aëronautics, the phase that interests us here most nearly, we should first take a broad view of the whole field to acquire the perspective necessary for a just appraisal of the value and of the limitations of mathematics in engineering. Mathematicians as investigators are concerned chiefly with but a single stage of the work. They cannot aid much in the first hardy adventures; they cannot help greatly in the final codifica-

tion—except as they strive as teachers better to prepare the freshman and the sophomore in his mathematical courses for the technical work to which he looks forward; they can help materially in the intermediate stage, provided they have schooled themselves in that style of mathematics which is suited to the treatment of engineering problems, that is in mechanics and mathematical physics.

2. In discussing aëronautical machines mathematically, or otherwise, a sharp distinction must be made between the lighter-than-air and the heavier-than-air types—the balloon and the plane types, as we may briefly designate them.

For the balloon type the first mathematical work will naturally be very elementary, dealing with buoyancy, with the expansion and contraction of gases owing to rise and fall in the level of flight and to changes of temperature. Arithmetic and algebra alone or the simplest portions of the calculus will suffice. If the coefficients of resistance to motion are known or assumed for different shapes of balloon, simple calculations will determine the power necessary for uniform rectilinear motion. As resistances are supposed to vary with some power, generally the square, of the velocity, easy problems in the calculus may be made regarding stopping and starting with no further use of differential equations than the "variables separable" type.

In discussing the airplane some small use of trigonometry is necessary. The pressure P in a plane surface of area S, which is taken as a first approximation to a wing, is assumed to be normal to the plane (the viscous tangential drag being neglected), and of the form $P = kSv^2 \sin i$, where i is the angle between the wing and the direction of motion. This is resolved vertically and horizontally to give the "lift" L and "drift" D as

$$L = kSv^2 \sin i \cos i, \qquad D = kSv^2 \sin^2 i.$$

In uniform horizontal flight $L = W$, the weight of the machine, and $D = F$, the thrust of the propeller. (To a first approximation the resistance of all surfaces other than the main wing of the monoplane or the pair of wings of the biplane are neglected.) The power necessary to drive the wing through the air is then nearly

$$\frac{Dv}{550} = \frac{W^2}{550kSv}.$$

This shows that for a given machine, the power is inversely as the velocity. The apparent anomaly is explicable by virtue of the fact that at higher velocities the machine flies at smaller angles of incidence i. A pretty exercise may be had by assuming that the resistance other than that due to the wings may be lumped together in a term Cv^2 added to the drift D so that the power becomes

$$\frac{W^2}{550kSv} + \frac{Cv^3}{550};$$

the minimum power may then be determined with its corresponding speed; and the further result that for the speed which minimizes the power, the weight per horsepower varies inversely as the speed.

There are a number of such simple applications of trigonometry and calculus which may be introduced in many a course in mathematics for the purpose of enlivening the work. One particularly interesting calculation is found by contrasting the expression for the power which has been obtained from the assumption (known to Euler) that the pressure varies with the sine of i with the expression which would be obtained from the erroneous assumption (generally attributed to Newton) that the pressure varies with the square of the sine of i. Those of you who are interested in the application of elementary mathematics to the airplane will find a great deal of suggestion in Painlevé and Borel's *L'Aviation*.

I wish next to mention those parts of advanced mathematics which are used by writers on aërodynamics. Beginning with Kirchhoff and Lord Rayleigh and reaching its culmination in recent work by Sir George Greenhill, the theory of functions of a complex variable has been indispensable in deriving theoretical formulas for the pressure P of a stream of air on a wing. Problems in hydrodynamics which take account of eddy (or vortical) motion or of internal friction (viscosity) are so difficult to solve that one may with reasonable accuracy say that for theoretical work in aërodynamics, eddies and friction must be disregarded; and for similar reasons the fluid motion is ordinarily restricted to plane motion. There are to be sure "end-effects" when a wing moves through the air, but owing to the length of span an approximation of value may be had by neglecting the "end-effects" and assuming that the motion of the air in all planes perpendicular to the edge of the wing is identical. Now by the method of "conformal representation" of the theory of functions the pressure exerted on a plane wing of various shapes, by the motion of the air, may in some cases be calculated, and the center of pressure may also be found.

These theoretical determinations of pressure and center of pressure are not precisely verified when experimentally measured in the wind tunnel and it is now customary to use the experimental values in place of those theoretically calculated. Nevertheless no student of theoretical aërodynamics can afford to be ignorant of the elements of hydrodynamics, including the theory of the reactions of streams on given contours, and this means that he must have some knowledge of the theory of functions of a complex variable, or rather let me say, some knowledge of the means of applying the principle of conformal mapping to the setting up of the functions which gives the map of one boundary and enclosed region upon another boundary and enclosed region. It is unhappily true that many extended and beautiful courses on the theory of functions leave the student wholly unable to carry through the applications necessary to the solution of these practical, or at least semi-practical, problems in hydromechanics. Might I suggest to all who teach the complex variable and the conformal map the possibility of drawing on uniplanar fluid motion as a source of instructive exercises?

3. Owing partly to the divergence between theory and experiment in fluid motion in simple cases and partly to the impossibility of obtaining a theoretical hydrodynamic solution for the motion of so complicated an object as an airship

or airplane, it is necessary in the main to treat the ship or plane as a rigid body to which are applied certain forces (resultant fluid pressures and moments of pressures) determined by wind tunnel experiments upon models. The problem of the motion of the machine then becomes one in the dynamics of a rigid body free to move in space (*i. e.*, with six degrees of freedom) subject to known forces. It is customary to refer the motion to axes fixed in the machine and moving with it. Now unfortunately moving axes are regarded as belonging to really advanced rigid dynamics. Routh in his two volumes *Elementary and Advanced Rigid Mechanics* leaves moving axes to the beginning of the Advanced Part. Single volume treatises such as Loney's *Dynamics of a Particle and of Rigid Bodies* are apt to stop before reaching moving axes. It is not at all unlikely that the present interest in the dynamics of flight will force us so to alter our texts and our courses as to include the theory of moving axes; for we cannot expect students to prolong their studies of mechanics unduly before entering upon the motion of aërial machines.

Whether referred to moving or fixed axes the equations of motion of an airplane or airship, with the forces represented by empirical equations experimentally determined, are too difficult to integrate without approximation. The first case treated is that of motion in a line nearly straight and inclined to the horizontal at a nearly constant angle. By considering the motion as made up of small departures from uniform flight in straight lines, the differential equations of the motion become linear equations with constant coefficients, and their solution is therefore a matter of well-known, though sometimes very tedious, routine. And right here allow me to intercalate the remark that we must not ourselves disdain arithmetic nor allow our students to be discouraged by it; there is current all too much of a feeling that only the beautiful general theories of mathematics are worthy the attention of real mathematicians and their students; the art and the science of getting a sufficiently accurate numerical solution of a numerical problem are of equal importance and dignity with general theory in all applications of mathematics.

The utility of the solution of the nearly uniform motion of the machine is found in the discussion of dynamical stability. The approximate equations are solved, as all linear equations with constant coefficients are solved, in terms of exponential functions with or without combination with trigonometric terms. If a machine dynamically stable is slightly disturbed from uniform motion the small departures from that standard state will die out as time goes on, *i. e.*, the exponential functions must have negative exponents like e^{-at}. It is a pretty and none too easy algebraic problem, solved in Routh's *Rigid Dynamics* but not in many books on algebra, to determine the conditions under which an algebraic equation has its real roots, and the real parts of its imaginary roots, negative. The practical importance of dynamical stability in a machine is that under normal atmospheric conditions and for short periods of time the machine may safely be flown "hands off" and the pilot has therefore some freedom to attend to other matters than the guiding of his craft.

I have been told that the theory of Bryan, recast by Bairstow, and supplemented by data of wind tunnel experiments, has enabled a machine to be designed and constructed which actually has been flown "hands off." If this be true it marks a great triumph for mathematical physics; whether it is true or not, we can confidently assert that to the English training in mathematical physics is in no small degree due the great and sudden advance in airplane design and the great success in aircraft warfare which have been realized in England in relatively few months.

4. I have shown you that the mathematics of aërodynamics leads from elementary algebra and arithmetic to the theory of functions of a complex variable and to the solution of linear differential equations with constant coefficients. I have shown how the theoretical work has come in that stage of the development of aëronautical engineering where it has been of real help in rapidly advancing the art. There are always with us branches of engineering and physics in which the right kind of mathematics is of great value for the rapid advance of those branches. This right kind of mathematics is the good old traditional Cambridge, England, type, the mathematics of Newton, of Green, of Maxwell, of Kelvin, of Rayleigh, a type of mathematics which in this country, owing perhaps to our preponderance of German-trained mathematicians, has all too little prestige. I sometimes wonder whether we do right to aim so exclusively at the continental type. It is worthy of note that our two great native mathematicians, George W. Hill and J. Willard Gibbs, were concerned with the applied side. May it not be that we in this country have such a natural bent toward the practical that a diligent cultivation of the British sort of mathematics would find a readier response among our students?

We are here not as research mathematicians but as teachers of collegiate mathematics. Our country has great industrial problems of peace and war to solve, and every one of us must help as he may. As we bend the mathematical twig, so will the tree incline. Let us without prejudice consider our curricula and with open mind introduce any necessary changes to make sure that the type of mathematics which we place before our students is that which will contribute most to the victory of our country in time of stress and to her prosperity in times of peace.

SPRING MEETING OF THE MINNESOTA SECTION.

The regular spring meeting of the Minnesota Section of the Mathematical Association of America was held at Carleton College on May 4, 1918. Seventy-eight people attended, including Edla G. Berger of College of St. Catherine, Father W. E. Etzel of St. Thomas College, C. H. Gingrich of Carleton College, chairman of he section, Jessie G. Quigley of College of St. Teresa, G. N. Bauer, W. H. Bus ey, R. R. Shumway, H. L. Slobin, R. M. Barton of the University of Minnesota, members of the Association. All in attendance at the meeting were, as the guests of Carleton College, most hospitably entertained through the day and at luncheon.

The program was arranged to give place to a paper from a subject closely allied to mathematics and it was the very good fortune of the section to be able to have Professor H. C. Wilson of Carleton College set before them with lantern slides the principal matter of his paper in the May number of *Popular Astronomy* upon the eclipse of the sun of June 8.

Professor Slobin presented a paper upon integration. He traced the development of the theory of integration corresponding to the development of the function concept, chronologically, from the work of Leibnitz and Newton through that of Cauchy, Dirichlet, Fourier, Riemann and Lebesgue, and established the relation of the Lebesgue integral to the preceding notions of integration.

Professor G. N. Bauer gave a talk upon the progress of the campaign for war savings stamps. He explained the meaning of the war savings stamp, its worth to the Government and to the purchaser and outlined the plan of the campaign, giving the amounts to be raised in the state and the method of reaching all people of the state.

It appeared to the program committee that it would be well to have some discussion of work which while not included in the regular text might be introduced in the class room at times. To this end Father Etzel was to discuss the quadratic involving a single parameter. Due to the fact that the early papers of the program had taken so much time it became necessary that Father Etzel simply present the subject of discussion and postpone the full discussion to the next meeting.

The Section wishes to coöperate in all possible ways with the Association and to this end arranged a discussion of the advisability of introducing the study of Descriptive Geometry into the college course in mathematics. This subject was discussed at the annual meeting of the Association with the general idea of having material on the program of direct interest in secondary school mathematics; it had the same purpose in the meeting of the section. This discussion was introduced by Professor W. H. Kirchner of the University of Minnesota. Professor Kirchner gave a historical outline of the study of the subject and the published text books; he showed how the subject could be a definite help to our present courses in mathematics and how the study would serve to clear up much of our present work which is difficult for the student to see, by giving him the power to visualize. He believes that if the subject were introduced into the present course in college mathematics, not as a distinct subject by itself but as related to other matter customarily treated, there would be a decided gain in clearness of understanding of mathematics and that the increased amount of time required for this work would not be appreciable. Professor Kirchner gave several illustrations of his discussion. The subject was discussed at considerable length by the section.

R. M. BARTON, *Secretary.*

SPRING MEETING OF THE MARYLAND–VIRGINIA–DISTRICT OF COLUMBIA SECTION.

The spring meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at Catholic University, Washington, D. C., on May 4, 1918. Among the thirty-two persons in attendance were the following members of the Association: O. S. Adams, U. S. Coast and Geodetic Survey; Clara L. Bacon, Goucher College; C. C. Bramble, U. S. Naval Academy; J. A. Bullard, U. S. Naval Academy; Paul Capron, U. S. Naval Academy; G. R. Clements, U. S. Naval Academy; A. Cohen, Johns Hopkins University; Alexander Dillingham, U. S. Naval Academy; Angelo Hall, U. S. Naval Academy; W. M. Hamilton, U. S. Nautical Almanac Office; Wm. E. Heal, U. S. Bureau of Plant Industries; L. S. Hulburt, Johns Hopkins University; W. W. Johnson, U. S. Naval Academy; A. E. Landry, Catholic University; Florence P. Lewis, Goucher College; Frank Morley, Johns Hopkins University; O. J. Ramler, Catholic University; H. M. Roeser, U. S. Bureau of Standards; R. E. Root, U. S. Naval Academy; W. F. Shenton, U. S. Naval Academy; C. E. Van Orstrand, U. S. Geological Survey and George Washington University.

The annual election of officers of the Section resulted as follows: Chairman, Professor A. E. Landry, Catholic University; Secretary-Treasurer, Professor R. E. Root, U. S. Naval Academy; Third member of the Executive Committee, Professor L. S. Hulburt, Johns Hopkins University.

The president, Professor A. Cohen, presided at both sessions of the meeting. The program was opened by a cordial and inspiring address of welcome by Rev. Dr. Edward A. Pace, of Catholic University. In the interval between the two sessions those attending were the guests of Catholic University at a most enjoyable luncheon, served in Graduates Hall.

Titles and authors of papers read were as follows:

"On the problem of elimination." Professor FRANK MORLEY, Johns Hopkins University.

"The doubly periodic functions connected with the curve $x^3 + y^3 = 1$." Mr. O. S. ADAMS, U. S. Coast and Geodetic Survey.

"Some results relating to the in- and circumscribed triangle of the rational quartic." Professor A. E. LANDRY, Catholic University.

"On Duhamel's theorem." Professor L. S. HULBURT, Johns Hopkins University.

"On the Missouri system of grading students." Professor FLORENCE P. LEWIS, Goucher College.

"A college training course in secondary mathematics." Dr. H. C. GOSSARD, U. S. Naval Academy.

"The use of polar line-coördinates." Mr. PAUL CAPRON, U. S. Naval Academy.

"A general system of approximate integration formulas." Mr. M. SASULY, Bureau of Standards (introduced by Mr. Harry M. Roeser).

"Matrices connected with the invariants of the equation of the second degree." Dr. J. A. BULLARD, U. S. Naval Academy.

RALPH E. ROOT, *Secretary.*

MEETING OF THE KENTUCKY SECTION.

The Tenth Annual Meeting of the Mathematics Section of the Kentucky Colleges and the Second Annual Meeting of the Kentucky Section of the Mathematical Association of America, was held at Georgetown College, Georgetown, Ky., May 11, 1918, in the Physics Lecture Room, Physics Building. The chairman, Prof. A. L. Rhoton, in a few words welcomed the members and visitors. The program with brief abstracts follows.

"Illustrated Lecture on Snowflakes." D. W. Martin, Professor of Physics, Georgetown College, assisted by Mr. C. V. Mullins.

Prof. Martin gave a brief account of the earliest work in snowflake photography by J. G. Greenough of Jericho, Vermont. Many slides were shown and attention was especially called to the pronounced hexagonal shape of the flakes, there being only one exception to this.

"Photogrammetry." Mr. V. G. Grove, University of Ky.

Photographic principles were first enunciated by Beautemps-Beaupré in 1791–1793. Following the invention of the sensitized plate, Colonel A. Laussedat in 1864 published the first work on photographic surveying. In 1865 A. Meydenbauer published applications to architecture. Guido Hauch then (1884) gave a graphical construction of a third perspective from two given ones. S. Finsterwalder laid the foundation of photogrammetry. Besides these, practical surveyors used photogrammetric methods to aid them in mapping mountainous regions. The two cases of one and two perspectives were discussed; the perspective being either vertical or inclined. The problems of orientation, graphical construction, reconstruction of an auxiliary figure, were given. An analytical expression for the coördinates of the space point in terms of the coördinates of the images was obtained. G. Hauch's construction of a third perspective was then discussed and it was shown that this subject has a very wide application.

"The Decipherment of Military Code Messages." H. R. Phalen, Berea College, Berea, Ky.

The speaker showed by some dozen large printed sheets the various methods of enciphering military code messages. All messages are divided into two great classes: transposition and substitution. In the first case the letters are simply rearranged according to some predetermined scheme, but each letter represents itself; that is, "a" means "a" and "k" means "k" wherever they are found. Consequently the vowel and consonant frequencies will be the same in this type of message as they are in any ordinary page of reading matter. Of this transposition type messages were presented in the simple vertical, diagonal, spiral, keyword and route cipher methods involving English, German, French and Spanish texts. The other great class is the substitution class where letters are

interchanged in their alphabetical position and where the vowel and consonant frequency tables are useless. Under this type the most interesting point was the construction of a military message by the method of the famous Playfair Cipher of the British army.

"A Generalization of the Mean Value Theorem." H. H. Downing, University of Ky.

The conditions of $0 < \theta < 1$ and $0 < h$ were removed and several functions were examined to find a relation between θ and h. For the function $ax^3 + bx^2 + cx + d$ we set $\varphi = \theta h$ and substitute the function in

$$\frac{f(x+h) - f(x)}{h} = f'(x + \varphi),$$

and obtain the equation

$$ah^2 - 3a\varphi^2 + (3ax + b)h - 2(3ax + b)\varphi = 0.$$

This is seen to be an ordinary hyperbola with transverse axis vertical and the hyperbola passing through the origin. (φ-axis vertical, H-axis horizontal.) The straight lines $\varphi = 0$, $\varphi = 1/3\ h$, $\varphi = 2/3\ h$, $\varphi = h$, $\varphi = 4/3\ h$, are the loci, respectively, of a focus, vertex, center, vertex and focus.

Prof. E. L. Rees was elected chairman for the year 1918–1919 and the present secretary was reëlected. The secretary's report was then given. Regret was expressed at the absence of Prof. W. H. Garnett, Wesleyan College, Winchester, Ky., and Prof. Henry Lloyd, Transylvania College, Lexington, Ky., both caused by illness. Regret was also expressed that Prof. H. R. Phalen was leaving the state.

Those present during the meetings were: A. L. Rhoton, David W. Martin, Georgetown College; G. C. Crooks, Center College; H. R. Phalen, Berea College, J. M. Maxey, Asbury College; P. P. Boyd, J. M. Davis, E. L. Rees, V. G. Grove, H. H. Downing, University of Kentucky.

H. H. Downing, *Secretary.*

BOOK NOTICES.

Send all communications about books to W. H. Bussey, University of Minnesota.

The number of books in the English language on the theory of functions of a complex variable continues to grow. One of the latest is "Functions of a Complex Variable," by Thomas M. Macrobert, Lecturer in Mathematics in the University of Glasgow. It is published by Macmillan and Co., London. The book is designed for beginners who have acquired a good working knowledge of the calculus. In order to make the subject not too difficult for beginners, the author has abstained from the use of strictly arithmetical methods and, while endeavoring to make his proofs sufficiently rigorous, he has based them mainly on geometrical conceptions.

"Has mathematics a realm apart from human life, fitting daily experiences in

places closely enough to be of use, but still not at all identical with it; or is it, indeed, the very same as the realm of human life? Is the differential equation only a refinement upon the real law of physics, the irrational only an approximation to the actual number of nature? Is the universe stable or will it some day disappear, wind its way back into chaos, leaving nothing but the truths of mathematics still standing? Is it true that chance does not exist really but only in seeming, or is everything purely chance, and are the laws of the universe merely the curves which we have drawn through a random few of an infinitely compact set of points? The consideration of these problems is what we mean by the phi osophy of mathematics." A new book on this subject has recently been published by the Open Court Publishing Company, 122 South Michigan Ave., Chicago, Illinois. It is entitled "Lectures on the Philosophy of Mathematics" and is by James Byrnie Shaw of the Department of Mathematics of the University of Illinois.

The number of trigonometry text books on the market is so very large that the author of every new one feels called upon to explain in his preface just why he has added to the list. The preface to the new "Plane and spherical trigonometry" by Professor Leonard M. Passano of the Massachusetts Institute of Technology says that the chief aim of the text are brevity, clarity and simplicity. The text "aims to present the trigonometry in such a way as to make it interesting to students approaching some maturity, and so as to connect the subject, not only with the mathematics which the student has already had, but also with the mathematics which, in many cases at least, is to follow." The book a ms to avoid the tendency to amplification which the author says trigonometry texts of late years have shown. The book is published by the Macmillan Company.

One of the most interesting chapters in the first course in calculus is the chapter on maxima and minima. The many practical problems that can be solved by the simple criteria given in the ordinary calculus text book interest the student exceedingly. They make him feel that the calculus has power. Mathematicians have always been interested in problems of maxima and minima. Those of the present day who are especially interested in the subject and who want a somewhat full treatment rather than the brief inadequate treatment given in the current text books will welcome the 193-page book on "The Theory of Maxima and Minima," by Professor Harris Hancock, of the University of Cincinnati, published by Ginn and Company.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

2719. Proposed by R. P. BAKER, University of Iowa.

Show that, $2x(\log x)^2 - x(x-1)(x+3)\log x + (x-1)^2(3x-1)$ is negative for $1 < x < \infty$.

2720. Proposed by CAPT. A. A. BENNETT, C.A.R.C., Galveston, Texas.

Given three points, A, B, C, in a plane, draw from an arbitrary fourth point D the segments AD, BD, CD. Also draw rays AA', BB', CC', making equal (small) angles respectively with

segments AD, BD, CD. The triangle determined by the three rays does or does not contain the point D according as the original triangle ABC does or does not contain D.

Prove the theorem, considering also the case in which A, B, C, D, are concyclic.

2721. Proposed by G. PAASWELL, New York City.

A, B are the termini of a horizontal line of length w. At a point C in this line, at a distance k from A, is applied a force P making an angle φ with the vertical. Determine the family of curves extending from A to B (and below AB) such that an equal normal distribution of loading p per unit length of curve around the periphery of the curve, will hold in equilibrium the force P. The parameter of the family so determined will be p.

2722. Proposed by FRANK IRWIN, University of California.

The number of terms in the general polynomial of the nth degree in m variables and in that of the mth degree in n variables is the same. It would be interesting to devise schemes which, without assuming this result, should exhibit the terms of these polynomials in one-to-one correspondence with each other.

2723. Proposed by GEORGE Y. SOSNOW, Newark, N. J.

The feet of the perpendiculars from the intersection of the diagonals on the sides of a cyclic quadrilateral M are joined to form a second quadrilateral N. Prove that N is a quadrilateral of minimum perimeter inscribed in M.

2724. Proposed by FRANK IRWIN, University of California.

Show that there is a unique set of real values, $x_1, x_2, x_3, \cdots, x_n$, that satisfy the equation

$$x_1^2 + x_2^2 + \cdots + x_n^2 - x_1x_2 - x_2x_3 - x_3x_4 - \cdots - x_{n-1}x_n - x_n + \frac{n}{2(n+1)} = 0.$$

2725. Proposed by S. A. COREY, Albia, Iowa.

Establish the identity,

$$(r_1r_2 + c_1r_3r_4 + c_2r_5r_6 + c_1c_2r_7r_8)(a_1a_2 + c_1a_3a_4 + c_2a_5a_6 + c_1c_2a_7a_8)$$
$$\equiv (r_9r_{10}r_{11}r_{12})^{1/2} + c_1(r_{13}r_{14}r_{15}r_{16})^{1/2} + c_2(r_{17}r_{18}r_{19}r_{20})^{1/2} + c_1c_2(r_{21}r_{22}r_{23}r_{24})^{1/2}$$

where $r_9 = a_1r_1 - c_1a_3r_2 - c_2a_5r_5 + c_1c_2a_7r_7$, $r_{10} = a_1r_2 - c_1a_3r_4 - c_2a_5r_6 + c_1c_2a_7r_8$, $r_{11} = a_2r_1 - c_1a_4r_3 - c_2a_6r_5 + c_1c_2a_8r_7$, $r_{12} = a_2r_2 - c_1a_4r_4 - c_2a_6r_6 + c_1c_2a_8r_8$, $r_{13} = a_3r_1 + a_1r_3 - c_2a_7r_5 - c_2a_5r_7$, $r_{14} = a_3r_2 + a_1r_4 - c_2a_7r_6 - c_2a_5r_8$, $r_{15} = a_4r_1 + a_2r_3 - c_2a_8r_5 - c_2a_6r_7$, $r_{16} = a_4r_2 + a_2r_4 - c_2a_8r_6 - c_2a_6r_8$, $r_{17} = a_3r_1 + c_1a_7r_3 + a_1r_5 + c_1a_3r_7$, $r_{18} = a_3r_2 + c_1a_7r_4 + a_1r_6 + c_1a_3r_8$, $r_{19} = a_6r_1 + c_1a_8r_2 + a_2r_5 + c_1a_4r_7$, $r_{20} = a_6r_2 + c_1a_8r_4 + a_2r_6 + c_1a_4r_8$, $r_{21} = -a_7r_1 + a_5r_3 - a_3r_5 + a_1r_7$, $r_{22} = -a_7r_3 + a_5r_4 - a_3r_6 + a_1r_8$, $r_{23} = -a_8r_1 + a_6r_3 - a_4r_5 + a_2r_7$, and $r_{24} = -a_8r_2 + a_6r_4 - a_4r_6 + a_2r_8$.

2726. Proposed by E. H. MOORE, The University of Chicago.

Let $\alpha_1(x, y)$, $\alpha_2(x, y)$, $\alpha_3(x, y)$, $\alpha_4(x, y)$, $\kappa_1(x, y)$, $\kappa_2(x, y)$ be six real-valued continuous functions of (x, y) over the unit-square S: $(0 \leqq x \leqq 1;\ 0 \leqq y \leqq 1)$. Let α_1, α_2, α_3, α_4 be symmetric functions and of positive type, i. e., for every real-valued continuous function $\xi(x)(0 \leqq x \leqq 1)$ $\int_0^1 \int_0^1 \xi(x)\alpha_\varrho(x, y)\xi(y)dxdy \geqq 0$. Prove the inequality:

$$(J_{15}'J_{26}''J_{37}'''J_{48}^{\text{iv}} - J_{16}'J_{28}''J_{35}'''J_{47}^{\text{iv}} - J_{25}'J_{46}''J_{17}'''J_{38}^{\text{iv}} + J_{26}'J_{48}''J_{15}'''J_{37}^{\text{iv}})\kappa_1\kappa_2\kappa_1\kappa_2 \geqq 0.$$

Here on the left there are four terms of which the first is

$$\int_0^1 \cdots \int_0^1 \alpha_1(u_1u_5)\alpha_2(u_2u_6)\alpha_3(u_3u_7)\alpha_4(u_4u_8)\kappa_1(u_1u_2)\kappa_2(u_3u_4)\kappa_1(u_5u_6)\kappa_2(u_7u_8)du_1 \cdots du_8;$$

thus the eight variables of integration $u_1 \cdots u_8$ are in order the eight arguments of the four functions $\kappa_1\kappa_2\kappa_1\kappa_2$ while the "integration" symbols J indicate how the variables are to be supplied as arguments to the four functions $\alpha_1\alpha_2\alpha_3\alpha_4$, e. g., J_{26}'' indicates that α_2 (the superscript'' determining the subscript 2) has the arguments u_2u_6.—Indicate another inequality of this type and determine the number of such inequalities.

SOLUTIONS OF PROBLEMS.

489 (Algebra). Proposed by S. A. COREY, Albia, Iowa.

Prove or disprove the following:

$$\begin{vmatrix} -x & -ay & -bu & abv \\ y & x & -bv & -bu \\ u & av & x & ay \\ -v & -u & y & -x \end{vmatrix}^2 + a \begin{vmatrix} x & -x & -bu & abv \\ y & y & -bv & -bu \\ u & u & x & ay \\ v & -v & y & -x \end{vmatrix}^2$$

$$+ b \begin{vmatrix} x & -ay & -x & abv \\ y & x & y & -bu \\ u & av & u & ay \\ v & -u & -v & -x \end{vmatrix}^2 + ab \begin{vmatrix} x & -ay & -bu & -x \\ y & x & -bv & y \\ u & av & x & u \\ v & -u & y & -v \end{vmatrix}^2 = \begin{vmatrix} x & -ay & -bu & abv \\ y & x & -bv & -bu \\ u & av & x & ay \\ v & -u & y & -x \end{vmatrix}^2.$$

II. SOLUTION BY A. M. HARDING, University of Arkansas.

The quantities X, Y, U, V, W, defined by the equations

$$\begin{aligned} xX - ayY - buU + abvV &= -xW, \\ yX + xY - bvU - buV &= yW, \\ uX + avY + xU + ayV &= uW, \\ vX - uY + yU - xV &= -vW, \end{aligned}$$

are proportional to the five determinants taken in order.

It can be easily shown that the last determinant in the left member of the proposed equation is equal to zero for all values of x, y, u, v, a, b. Hence the above equations may be written in the form

$$\begin{aligned} (X+W)x - aYy - bUu &= 0, \\ Yx + (X-W)y - bUv &= 0, \\ Ux + (X-W)u + aYv &= 0, \\ Uy - Yu + (X+W)v &= 0. \end{aligned}$$

The quantities x, y, u, v, can satisfy this system of homogeneous linear equations if, and only if, the determinant

$$\begin{vmatrix} X+W & -aY & -bU & 0 \\ Y & X-W & 0 & -bU \\ U & 0 & X-W & aY \\ 0 & U & -Y & X+W \end{vmatrix}$$

is equal to zero. It can be shown that the value of this determinant is k^2, where

$$k = (X^2 - W^2) + aY^2 + bU^2.$$

Hence $X^2 + aY^2 + bU^2 = W^2$, or, since $V = 0$, $X^2 + aY^2 + bU^2 + abV^2 = W^2$. That is, the relation stated in the problem holds for all values of x, y, u, v, a, b.

NOTE. We are publishing a second solution of this problem for two reasons: First, because the solution above is essentially different from the one published in the March number; and second, because the conclusion drawn in that solution that the identity does not always exist is incorrect. EDITORS.

271 (Number Theory). Proposed by HORACE OLSON, Chicago, Illinois.

Prove that if x, y, z, u, v, and w are integers such that $x^2 + y^2 = u^2$, $x^2 + z^2 = v^2$, $y^2 + z^2 = w^2$, then the product $xyzuvw$ is divisible by 518400.

Solution by the Proposer.

I shall first prove some lemmas.

1) If x, y, and u are integers such that $x^2 + y^2 = u^2$, then either x or y is divisible by 3. This follows from the fact that any perfect square is congruent, modulo 3, to either 0 or 1.

2) If x, y, and u are integers such that $x^2 + y^2 = u^2$, then either x or y is divisible by 4; for any perfect square is congruent, modulo 16, to 0, 1, 4, or 9.

3) If x, y, and u are integers such that $x^2 + y^2 = u^2$, then at least one of the numbers x, y, u is divisible by 5; for any perfect square is congruent, modulo 5, to 0, 1, or 4.

Hence, from the hypotheses, at least two of the numbers x, y, z are divisible by 3. If the notation be so chosen that these are x and y, the first equation shows that u also is divisible by 3. We can then divide both terms of this equation by 9; it then follows that either $x/3$ or $y/3$ is divisible by 3. Thus the product $xyzuvw$ is divisible by 3^4. In the same way it can be proved that the product is divisible by 4^4.

By lemma 3, at least one of the numbers x, y, u is divisible by 5. Similarly, at least one of each of the sets x, z, v and y, z, w is divisible by 5. Thus at least two of the numbers x, y, z, u, v, w are divisible by 5.

Therefore the product $xyzuvw$ is divisible by $3^4 \cdot 4^4 \cdot 5^2$, or 518400.

Also solved by H. C. Feemster and J. L. Riley.

2661. Proposed by Artemas Martin, Washington, D. C.

Find a parallelepipedon whose edges, and the diagonals of its faces, are all rational whole numbers.

Solution by the Proposer.

Denote the edges by x, y, and z; then

$$x^2 + y^2 = \square, \quad x^2 + z^2 = \square, \quad y^2 + z^2 = \square. \tag{1, 2, 3}$$

Assume $x = 2pq$, $y = p^2 - q^2$; then by substitution,

$$x^2 + y^2 = (2pq)^2 + (p^2 - q^2)^2 = (p^2 + q^2)^2,$$

a square;

$$x^2 + z^2 = (2pq)^2 + z^2, \quad y^2 + z^2 = (p^2 - q^2)^2 + z^2, \tag{4, 5}$$

which must be made squares.

Put

$$(2pq)^2 + z^2 = \left(z - \frac{2pqr}{s}\right)^2,$$

which gives

$$z = \frac{pq(r^2 - s^2)}{rs}.$$

Substituting in (5) the value of z found above,

$$(p^2 - q^2)^2 + \frac{p^2q^2(r^2 - s^2)^2}{r^2s^2} = \square,$$

or

$$r^2s^2(p^2 - q^2)^2 + p^2q^2(r^2 - s^2)^2 = \square. \tag{6}$$

Expanding (6) it may be written

$$(p^2r^2 + q^2s^2)(q^2r^2 + p^2s^2) - 4p^2q^2r^2s^2 = \square,$$

which will be the case when

$$q^2r^2 + p^2s^2 = 4q^2s^2. \tag{7}$$

By transposition, (7) becomes

$$q^2r^2 = 4q^2s^2 - p^2s^2 = s^2(4q^2 - p^2) = s^2\left(\frac{mp}{n} - 2q\right)^2,$$

say; and we get

$$\frac{p}{q} = \frac{4mn}{m^2 + n^2}, \qquad \frac{r}{s} = \frac{2(m^2 - n^2)}{m^2 + n^2};$$

$$x = 2pq = 8mn(m^2 + n^2), \qquad y = p^2 - q^2 = (4mn)^2 - (m^2 + n^2)^2,$$
$$z = \frac{pq(r^2 - s^2)}{rs} = \frac{2mn[4(m^2 - n^2)^2 - (m^2 + n^2)^2]}{m^2 - n^2}.$$

Reducing the foregoing values of x, y, z to a common deominator $m^2 - n^2$ and then discarding it, we finally have

$$x = 8mn(m^2 - n^2)(m^2 + n^2),$$
$$y = (m^2 - n^2)[(4mn)^2 - (m^2 + n^2)^2],$$
$$z = 2mn[4(m^2 - n^2)^2 - (m^2 + n^2)^2].$$

Take $m = 2$, $n = 1$ and we get $x = 240$, $y = 117$, $z = 44$, the smallest values known. The diagonals of the faces are 267, 244, 125.

Also solved by C. B. HALDEMAN and L. E. LUNN.

2664. Proposed by J. W. NICHOLSON, Baton Rouge, La.

Find the sum of the series, $\frac{1}{3} - \frac{2}{15} + \frac{3}{35} - \cdots + (-1)^{n+1}\frac{n}{(2n-1)(2n+1)}$.

SOLUTION BY S. W. REAVES, University of Oklahoma.

The exponent of -1 in the general term should obviously be $n + 1$. Making this correction, the general term may be written,

$$(-1)^{n+1}\frac{n}{(2n-1)(2n+1)} = \tfrac{1}{2}(-1)^{n+1}\left(\frac{n}{2n-1} - \frac{n}{2n+1}\right).$$

Let S denote the required sum. Then

$$S = \frac{1}{2}\left[\sum_{i=1}^{n}(-1)^{i+1}\frac{i}{2i-1} + \sum_{i=1}^{n}(-1)^{i+2}\frac{i}{2i+1}\right]$$
$$= \frac{1}{2}\left[\left(1 - \tfrac{2}{3} + \tfrac{3}{5} - \cdots + (-1)^{n+1}\frac{n}{2n-1}\right) + \left(\tfrac{1}{3} + \tfrac{2}{5} - \cdots + (-1)^{n+2}\frac{n}{2n+1}\right)\right]$$
$$= \frac{1}{2}\left[1 - 1 + 1 - 1 + \cdots + (-1)^{n+2}\frac{n}{2n+1}\right].$$

If n be even, $S = \dfrac{n}{4n+2} = \frac{1}{4}$ when n is infinite.

If n be odd, $S = \dfrac{n+1}{4n+2} = \frac{1}{4}$ when n is infinite.

Also solved by BANCROFT H. BROWN, E. H. WORTHINGTON, E. B. ESCOTT, GEORGE F. WILDER, PAUL CAPRON, HORACE OLSON, ELIJAH SWIFT, and ARNOLD DRESDEN.

2665. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

A telegraph wire, which weighs 1/10 lb. per yard, is stretched between poles on a level ground so that the greatest dip of the wire is 3 feet. Find, approximately, the distance between the poles when the tension at the lowest point of the wire is 140 pounds.

SOLUTION BY G. PAASWELL, New York City.

MacMahon's *Hyperbolic Functions*, pages 47 to 51 discusses the two cases here, (*a*) when the wire is taken as inextensible, and (*b*) as extensible. Under (*a*) $y/c = \sec\varphi$; $x/c = gd^{-1}\varphi = \log_e \tan(\pi/4 + \varphi/2)$. The origin is the point midway between the poles and a distance c below the lowest point of the wire. Use the yard as the unit of length. $c = H/w = 140/0.1 = 1400$. $y = 1 + c = 1401$. Whence, $\varphi = 2^\circ\ 9'\ 54''$. Thence, $x = 52.913$ and since $L = 2x$, the re-

quired length is 105.826 yards. Note here that if the weight is taken as distributed along the horizontal, i. e., along the x-axis instead of along the wire there is obtained by simple moments $c\,dy = x\,dx$ and with the origin as before except that the low point is taken instead of a point c below, there is obtained the parabolic relation $x = \sqrt{2cy}$, with $y = 1$. Hence, $x = \sqrt{2800} = 52.915$ and $L = 105.83$ yds., which is sufficiently close for this problem. As a matter of interest, it is seen that since $y/c = \cosh x/c$, the origin a distance c below the low point, by expanding the cosh to two terms, $1 + x^2/2c^2$, and replacing y by $y - c$ the parabolic relation above is obtained, which is a measure of the approximation of this method of solving the catenary.

(*b*) Here $y/c = \sec\varphi + \frac{1}{2}m\tan^2\varphi$; $x/c = gd^{-1}\varphi + m\tan\varphi$. $m = 1H$, where 1 is the unit extension of the wire. Since the wire weighs 0.1 pound per yd., taking steel as weighing 10.2 pounds per sq. in. per yd., and the modulus of elasticity as 30,000,000 pounds per sq. in., we have since $1 = H/AE$, where a is the area of the section of the wire and E the modulus, $m = 6664 \times 10^{-6}$. Expressing the y equation as a quadratic in sec φ and solving for φ, $\varphi = 2°\ 5'\ 54''$. Substituting this value in the equation for x and again replacing the gudermanian by its logarithmic equivalent we get $x = 54.74$ and $L = 109.58$ yds. Do not mistake the difference between this value and the above as the extension of the wire. Its meaning is that when allowance is made for extension it is necessary to spread the supports above the difference apart in order to maintain the deflection of one yard. The parabolic expression for this case is more complicated than the exact.

Also solved by J. V. BALCH, HORACE OLSON, A. R. NAUER, PAUL CAPRON, and LOUIS G. POOLER.

2666. Proposed by W. WOOLSEY JOHNSON, Annapolis, Md.

Ten equations between five quantities, x_1, x_2, x_3, x_4, x_5 being written as follows: $x_1 = 1 - x_2x_4$ and four others formed by the cyclic interchange of the suffixes; also $x_5x_1x_2 = x_5 + x_2 - 1$ and four others formed by the cyclic interchange; prove that only three of these equations are independent. In other words, the values of x_1 and x_2 being assumed at pleasure, x_3, x_4, and x_5 can be so determined as to satisfy all ten equations.

I. SOLUTION BY ARNOLD DRESDEN, University of Wisconsin.

Denote by E_i the function $x_i + x_{i+2}x_{i+3} - 1$ and by $\overline{E}_i$ the function $x_ix_{i+1}x_{i+2} - x_i - x_{i+2} + 1$. Then the proposed equations may be written in the form $E_i = 0$, $\overline{E}_i = 0$, $(i = 1, \cdots, 5)$, the subscripts being reduced modulo 5.

We have the following identities:

$$(1) \qquad \begin{cases} x_{i+1}E_i - E_{i+3} \equiv \overline{E}_{i+1} \\ x_{i-1}E_i - E_{i+2} \equiv \overline{E}_{i+2}, \end{cases}$$

so that the equations $\overline{E}_i = 0$ are dependent upon the equations $E_i = 0$.

But from the relation (1), we derive, moreover, the following

$$(5) \qquad E_{i-1} + x_{i-1}E_i \equiv x_{i+2}E_{i+1} + E_{i+2}.$$

Hence from any three of the equations, $E_i = 0$, the remaining two must follow.

II. SOLUTION BY HORACE L. OLSON, Heidelberg University, Tiffin, Ohio.

The ten equations mentioned are

$$(1)\ x_1 = 1 - x_3x_4 \qquad (6)\ x_5x_1x_2 = x_5 + x_2 - 1$$

$$(2)\ x_2 = 1 - x_4x_5 \qquad (7)\ x_1x_2x_3 = x_1 + x_3 - 1$$

$$(3)\ x_3 = 1 - x_5x_1 \qquad (8)\ x_2x_3x_4 = x_2 + x_4 - 1$$

$$(4)\ x_4 = 1 - x_1x_2 \qquad (9)\ x_3x_4x_5 = x_3 + x_5 - 1$$

$$(5)\ x_5 = 1 - x_2x_3 \qquad (10)\ x_4x_5x_1 = x_4 + x_1 - 1.$$

Let us assume, first, that $x_1x_2 \neq 1$. Then from equations (4), (2), and (3), respectively, we find that,

$$x_4 = 1 - x_1x_2, \qquad x_5 = (1 - x_2)/(1 - x_1x_2), \qquad x_3 = (1 - x_1)/(1 - x_1x_2).$$

To show that the remaining equations are dependent upon these three, we note that equations (3), (4), (2) lead respectively to

$$1 - x_2x_3 = 1 - x_2 + x_1x_2x_5, \qquad 1 - x_2x_3 = 1 - x_2 + x_5(1 - x_4), \qquad 1 - x_2x_3 = x_5.$$

This last equation is equation (5).

Similarly we have

$$1 - x_3x_4 = 1 - x_4 + x_4x_5x_1, \qquad 1 - x_3x_4 = 1 - x_4 + x_1 - x_1x_2, \qquad 1 - x_3x_4 = x_1,$$

This last equation is (1).

Multiplying (4) by x_5, we have $x_5x_1x_2 = x_5 - x_5x_4$ and then $x_5x_1x_2 = x_5 + x_2 - 1$ from (2). This last equation is (6). Since, then, equations (2) to (5) and (7) to (10) are obtained from (1) and (6) by cyclic permutation of the suffixes, it follows that upon equations (2), (3), and (4) all the others depend. This, we note, is true whether or not $x_1x_3 = 1$. If $x_1x_2 = 1$, we find from (4), (1), and (2), respectively, $x_4 = 0$, $x_1 = 1$, $x_2 = 1$. Equations (3) and (5) then both become $x_3 + x_5 = 1$. Under these conditions equations (6) to (10) are satisfied identically, and x_3 and x_5 are not determined.

Also solved by ELIJAH SWIFT, PAULINE SPERRY, C. C. YEN, and HERBERT N. CARLETON.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas.

DISCUSSIONS.

I. CONCERNING DIRECTION COSINES AND HESSE'S NORMAL FORM.

By MAXIME BÔCHER, Harvard University.

For many years I have felt that Hesse's form for the equation of a line is out of place in a course on elementary analytic geometry[1] and have omitted it from my own course. I am glad to find the same view expressed by two writers in recent numbers of the MONTHLY (p. 476, 1917 and p. 181, 1918) who give proofs of the formula for the distance from a point to a line which do not involve the use of Hesse's form. Perhaps I may be permitted to call attention to a proof of this kind which I have published elsewhere[2] and which seems to me to have two advantages over those recently given in the MONTHLY, namely, *first*, that it requires no reference to the figure with the consequent necessity (if we wish to be complete, that is, really to give a *proof*) of examining exhaustively the various forms the figure may take on; and *secondly*, that it admits of immediate extension to the corresponding problem in solid analytic geometry.

It is perhaps not quite so clear that Hesse's form for the equation of a plane in solid analytic geometry should be removed from the pages of elementary text books or relegated to small-type sections, since this form is commonly used not merely in connection with the formula for the distance from a point to a plane but also as the primary form of the equation of a plane from which the other forms are deduced. I will indicate how I am accustomed to dispense with it altogether, and at the same time to relegate the too pervasive direction cosines

[1] It has its definite, though very limited, place in higher work.

[2] Bôcher, *Plane Analytic Geometry*, Henry Holt & Co., 1915, p. 38.

of the ordinary treatment to the subordinate position in which I think they belong.

The direction of a line in space is determined by the ratios $l : m : n$ of the projections of any segment of this line on the coördinate axes. These ratios are called the *direction ratios* of the line. They are the natural generalization of the slope of a line in plane analytic geometry. These direction ratios may, of course, be regarded as the ratios of the direction cosines,[1] and, conversely, the direction cosines may be expressed in terms of the direction ratios by means of the formulæ

$$(1)\quad \cos\alpha = \frac{l}{\sqrt{l^2+m^2+n^2}}, \quad \cos\beta = \frac{m}{\sqrt{l^2+m^2+n^2}}, \quad \cos\gamma = \frac{n}{\sqrt{l^2+m^2+n^2}}.$$

If two lines have the direction ratios $(l_1 : m_1 : n_1)$ and $(l_2 : m_2 : n_2)$ respectively, the condition for parallelism is

$$l_1 : m_1 : n_1 = l_2 : m_2 : n_2,$$

and the condition for perpendicularity is

$$l_1l_2 + m_1m_2 + n_1n_2 = 0.$$

The line connecting the points (x_1, y_1, z_1) and (x_2, y_2, z_2) evidently has as its direction ratios $(x_2 - x_1 : y_2 - y_1 : z_2 - z_1)$.

By the direction ratios of a plane we understand the direction ratios of its normals.

It now easily follows that the equations of the line through (x_1, y_1, z_1) with direction ratios $(l : m : n)$ are

$$x - x_1 : y - y_1 : z - z_1 = l : m : n,$$

which, if none of the quantities l, m, n are zero may be written

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.$$

On the other hand, if we have a plane through (x_1, y_1, z_1) with direction ratios $(l : m : n)$, the point (x, y, z) lies in this plane when and only when the line connecting it with (x_1, y_1, z_1) is perpendicular to the normals to the plane. Consequently the equation of the plane is

$$l(x - x_1) + m(y - y_1) + n(z - z_1) = 0.$$

From this it is easy to prove that every equation of the first degree

$$Ax + By + Cz + D = 0$$

represents a plane whose direction ratios are $(A : B : C)$. The further formulæ concerning lines and planes are now easily obtained.

[1] Except in the case of isotropic lines which have no direction cosines but do have direction ratios.

The method of presentation here sketched not only dispenses with Hesse's normal form entirely, but is to my mind simpler and more appropriate than the one ordinarily given. Direction cosines can be introduced by means of (1) whenever they are wanted, but there are very few occasions where there is any advantage in using them, the formula

$$\cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2$$

for the cosine of the angle between two lines being one of the few formulæ which are a little simpler when direction cosines are used than when written in terms of direction ratios.

As an introduction to more advanced work the method here suggested has the advantage of being applicable to isotropic lines and planes as well as to others.

II. Concerning a Method of Constructing the Graph of an Equation in Which the Variables May Be Separated.

By E. L. Rees, University of Kentucky.

If an equation can be put in the form $\varphi(x) = \psi(y)$, *i. e.*, if the variables can be separated, the graph may be constructed in the following simple manner: Construct the graphs of the equations $y' = \varphi(x)$, $x' = \psi(y)$. Draw an ordinate of the first curve and an abscissa of the second curve equal to the ordinate of the first. The intersection of this ordinate and abscissa is a point on the required curve. In this manner any desired number of points may be rapidly constructed.

The following theorems[1] giving the properties of the graph as determined from those of the auxiliary curves are easily deduced. Indeed, the truth of most of them may be inferred from geometric intuition.

1. If x' and y' are both increasing or both decreasing functions in corresponding intervals the curve will rise in those intervals.

2. If one function increases and the other decreases the curve will fall in the corresponding intervals.

3. A vertical line of symmetry of the first or a horizontal line of symmetry of the second auxiliary curve is a line of symmetry of the required curve.

4. A turning point (or a multiple turning point) on either auxiliary curve in general corresponds to a turning point (or multiple turning point) on the required curve.

5. Two corresponding turning points give a node if both are maximum or both minimum, or a conjugate point if one is maximum and the other minimum.

6. A double turning po nt and a corresponding turning point give a cusp.

7. A triple turning point and a corresponding turning point if both are maximum or both minimum give a point of osculation.

[1] In the statement of these theorems it is to be understood that turning points and multiple turning points on the second curve are those points at which the tangent is vertical, while these terms as applied to the first curve have their usual meaning.

Other theorems of a similar nature could be added. These theorems may be illustrated by the following simple equations, the numbers in the parentheses indicating the theorems which the equations are intended to illustrate. $y^2 - 1 = x^3 - 3x$, $y^2 - 1 = x^3$(1, 2, 3, 4); $y^2 - 2 = x^3 - 3x$, $y^2 + 2 = x^3 - 3x$ (5); $y^2 = x^3$ (6); $y^2 = x^4 - x^6$ (7).

It is, of course, true that the advantages of this method of constructing graphs are greatest for the more complicated forms of equations.

This principle may also be applied in the construction of graphs of parametric equations. For, if the parametric equations are $x = \varphi(t)$, $y = \psi(t)$ we may use as our auxiliary equations $x = \varphi(y')$, $y = \psi(x')$ and proceed as before.

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY R. C. ARCHIBALD, Brown University, Providence, R. I.

In so far as information is obtainable the December issue of this MONTHLY will contain a directory of undergraduate mathematics clubs which have been in operation during at least a part of 1917–18. Certain statistics, and general discussion of club interests, will be introduced also. It is hoped that those clubs which have either failed to discover themselves to the editor or neglected to reply to his repeated solicitations for information, will report to him as soon as possible and, in any case, not later than October 6.

CLUB ACTIVITIES.

MATHEMATICS CLUB OF IOWA STATE TEACHERS COLLEGE, Cedar Falls, Iowa.

This club was organized in December, 1909, "to further the work in mathema'ics in the college and in the state." The officers for 1917–18 were: President, Ira S. Condit, head of the department of ma hematics; secretary, Irma Hemphill '18; executive committee, Professor Peter Luteyn and Grace Hillier '18, who prepare programs and decide on dates of meetings The programs between June, 1916, and March 1918, were as follows:

June 28, 1916: "Teaching factoring" by Professor Robert D. Dougherty.

July 12: "Standard test;" by Professor E. E. Watson, of Parsons College.

August 2: "The problem and the process of analysis in arithmetic" by Professor Condit.

August 16: "Ratio and proportion," by Professor Charles W. Wester of Iowa State College and Rodney W. Babcock, instructor in mathematics at the University of Wisconsin.

October 18: "The place of mathematics in education" by Professor Wester.

November 15: Report of Committee on Elimination, of Iowa State Teachers Association, by Ruth Smith '18 and Tracey Hodson '18.

April 24, 1917: "Methods of statistical study" by Professor Wester.

May 16: Conservation in mathematics—"Conservation of time and energy by the development of algebraic symbolism" by Irma Hemphill '18; "The

metric system" by Alice O'Connor '18; "Short cuts in arithmetic" by Mae Howell '18; "Short cuts in high school mathematics" by Doris de Bar '18; "Conservation in geometry" by Ruth Smith '18.

July 11: "Elimination in mathematics" by Professor Emma L. Lambert.

July 25: "Elimination in mathematics" by Professor Luteyn.

March 6, 1918: "Contents of a course in arithmetic" by Professor Wester.

JUNIOR MATHEMATICS CLUB, University of Minnesota, Minneapolis, Minn.

The first meeting of this club was held on December 9, 1914. All students who have completed the course in integral calculus, and seniors who are majoring in mathematics, are eligible for membership. The staff of instruction in the department of mathematics and astronomy is also invited to attend the meetings of the club. The number of its members was 20 last year and the average attendance at meetings was 15.

Officers, 1917–18: President, Jemima Olson '18; secretary, Elizabeth Carlson Gr. These officers constitute the program committee also.

November 8, 1917: "The drawing of a circle having five times the area of a given circle" by Jemima Olson '18.

November 22: "How to draw a straight line without the use of a ruler" by Adelina Anderson '18.

December 6: "The Russian peasant method of multiplication" by Elma F. Hario '18.

December 13: Joint meeting with the University Mathematical Club[1]—"A generalized form of mathematical induction" by Professor William H. Bussey; "A note on the eight queens problem"[2] by Professor Bussey.

January 17, 1918: "The reduction of a quadratic form to the sum of squares" by Vera Wright Gr.

March 7: "Magic squares" by Hildegarde Swenson '18.

March 28: "Wallace lines" by Roger A. Johnson, Professor of Mathematics at Hamline University, St. Paul, Minn.

April 10: "Calculating machines" by Ralph M. Barton, instructor in mathematics, and Professor Bussey.

April 18: Annual banquet.

April 24: "The educational value of the history of mathematics" by William O. Beal, assistant astronomer to the university.

MATHEMATICS CLUB OF MOUNT HOLYOKE COLLEGE, South Hadley, Mass.

This club, which has just completed the tenth year of its existence, was founded for the purpose of "presenting to its members a broader view of mathematics." Last year it had 38 active and 15 associate members, the average attendance at

[1] This club is composed of faculty members and graduate students. Its programs are usually of too advanced a character for members of the Junior Mathematics Club.

[2] *Cf.* W. W. R. Ball, *Mathematical Recreations and Essays*, fifth edition, London, Macmillan, 1911, pp. 113–118.

meetings being 20. Junior and senior students who are doing major work in mathematics are eligible for active membership and such students become members by signing the constitution. Any sophomore who has elected mathematics as one of her major subjects may become an associate member by a majority vote of the members present at any meeting. Any member of the staff of the mathematics department and any graduate student in mathematics is eligible for either active or associate membership. Only active members may vote or hold office. Active members are charged a tax of fifty cents a year.

Officers 1917–18: President, Alice Weeks '18; vice-president, Evelyn Clift '19; secretary-treasurer, Jeannette Bickford '18; executive board: Ruth Carpenter '18, and Professor Emilie N. Martin.

October 21, 1916: Social meeting; the club was entertained by guessing various mathematical puzzles.

November 17: "Mathematical theory of probabilities and its application to games of chance" by Ruth Carpenter '18; "Magic squares" by Alice Weeks '18.

December 16: "Poincaré" by Helen Hughes '17; "Fermat, Fermat's Theorem, and Painlevé" by Florence Allen '17; an article on Sonja Kovalevski read by Ethel Anderson '19.

January 16, 1917: A talk on the models of surfaces owned by the college, and an exhibition of photographs of snow crystals by Professor Martin.

February 17: "Linkages" by Margaret Wilcox '19 and Evelyn Clift '19 (illustrated by models which they had constructed for the occasion); selections from Cassius Keyser's "Essays and Addresses" [1] read by Ruth Hemenway.

March 24: "Continued fractions" by Jeannette Bickford '18.

April 21: "Photogrammetry and its use in the present war" by Charles L. Bouton, associate professor of mathematics at Harvard University.

May 22: Election of officers for 1917–18.

In 1917–18 the number of meetings was reduced to five on account of war-work in which the members were engaged.

September, 1917: Social meeting to receive new members.

November: "Training in mathematics in a Russian artillery school" by Mr. Alexander Pell of South Hadley.

February 16, 1918: "Mathematics in the present war" by Captain Peter F. Field, professor at the University of Michigan.

March 9: "Applied mathematics in the laboratory work of a great electric company" by Agnes Eastman '13.

May 25: "Mathematics of insurance" by Helen Hughes '17; election of officers for 1918–19.

THE MATHEMATICAL CLUB OF THE UNIVERSITY OF NEBRASKA, Lincoln, Neb.

This club was organized in 1915 "to stimulate the mathematical interests of its members by presenting illuminating and varied phases of mathematics in

[1] The selections were taken from two essays: "The human worth of rigorous thinking" and "The human significance of mathematics."

pure and in applied form." According to the preliminary organization circular: "Many absorbing topics that can hardly be dealt with in the class room will find a natural place in this club. It is hoped that the members will gain a new and valuable insight into the beauty and wealth of the mathematical realm and a greater knowledge of the many ways in which mathematics touches life."

The charter members were recommended by the faculty. Thereafter members have been elected by the club on the recommendation of its executive committee which, as a matter of established custom, first consults the faculty. "As a rule the faculty recommends to membership those students who have shown distinction in their first year mathematics courses (through analytic geometry)"

Officers, 1917–18: President, Julia L. Torrence '18; vice-president, Eimo G. Funke '19; secretary-treasurer, Frances Botkin '19; faculty-adviser, Professor Henry Blumberg. These officers constituted the executive committee.

The club meets once a month, 7.30–9 00 p.m.; the number of members in attendance has varied from 20 to 60, with an average of about 35. All members are expected to contribute to the topics of the evening. Interesting problems are posted on the club's bulletin board and these problems are later discussed at the meetings. During 1916–17 three prizes were awarded by the club, mainly for activity in connection with the bulletin board. The programs for 1915–18 have been as follows:

October 14, 1915: Organization meeting; talks by different members of the faculty.

November 11: "Various definitions of π" by James H. Taylor '16; "The game of nim" by Professor Blumberg.

December 9: "Some problems from analysis situs" by Edward M. Kadlack '16; "Selected facts from the theory of numbers" by Professor William C. Brenke.

January 13, 1916: Social meeting, log fire, refreshments, everybody expected to bring an anecdote about a famous mathematician.

February 10: "Cantor's famous contribution to the study of the infinite" by Professor Blumberg; "The cycloid" by Herbert Grumman '15.

March 9: "Euclid's fifth postulate" by Olive Bayles '16; "Soap bubbles and mathematics" by Professor Oliver Gish.

April 13: "Lightning calculators" by Ezra Andreson '18; "The method of Archimedes" by Professor Albert L. Candy.

May 11: "Some unsolved problems in mathematics" by Professor Ellery W. Davis; social hour.

October 12: Election of officers; social hour.

November 9: "Summation of certain series" by Walter F. Weiland '18; "History of perspective drawing" by Professor Blumberg.

December 14: "Certain problems of interpolation" by Professor W. C. Brenke; bulletin board problems.

January 11, 1917: Social meeting; mathematical games; everybody expected to bring a geometric, algebraic or logical paradox.

February 15: "History of logarithms" by William F. Joachim '18.

March 15: "Selected topics from the theory of probability" by Professor Davis; bulletin board problems.
April 15: "The roots of unity" by Alva L. Sikes '19; "History of the Arabic numerals" by Professor Candy.
October: Election of officers; social hour.
November 9: "Selected topics from projective geometry" by Frances Botkin '19; "Mathematical curricula in different countries" by Professor Candy.
December 13: "Magic squares" by Josefa Seely '18; "Graphical methods" by Professor Lulu Runge.
March 14, 1918: Bulletin board problems; social hour with mathematical games.
April 11: "Selected facts from the theory of numbers" by Professor Blumberg; bulletin board problems; mathematical games.
May 9: "Development of mathematical symbols" by Bernice Downing '18; bulletin board problems; social hour with mathematical games and mathematical conundrums.

THE MATHEMATICS CLUB OF THE UNIVERSITY OF OKLAHOMA, Norman, Okla.

At a meeting on October 26, 1916, a group of students, who had decided to organize a mathematics club, appointed a committee, with Professor Samuel W. Reaves, head of the department, as chairman, to draw up a constitution. This constitution, adopted at the following meeting, was as follows:

PREAMBLE.

We, the undersigned, appreciating the advantages to be derived from an association which shall give opportunity for the presentation and discussion of mathematical subjects of interest, do hereby organize ourselves into a mathematical club, and we agree to be governed by the following Constitution.

ARTICLE I. *Name.*

Section 1. This association shall be called The Mathematics Club of the University of Oklahoma.

ARTICLE II. *Members.*

Section 1. Membership in this Club is limited to such students and teachers in the University of Oklahoma as are interested in the subject of mathematics.

Section 2. Proposals for membership shall be in writing and may be submitted at any regular meeting.

Section 3. Voting upon members shall be by ballot, a majority vote being necessary for election.

ARTICLE III. *Officers.*

Section 1. The officers of this club shall be a President, a Vice-President, and a Secretary-Treasurer, which officers shall be elected from among the students majoring in mathematics.

Section 2. Officers of the club shall be elected by ballot at the first regular meeting of each semester.

Section 3. Each officer shall serve until his successor is duly elected. Vacancies may be filled temporarily by presidential appointment until the next regular meeting of the club.

Section 4. The three officers mentioned in Section 1, together with one faculty member selected by the faculty members of the club, shall constitute the Program Committee.

ARTICLE IV. *Meetings.*

Section 1. Regular meetings of the club shall be held on the second and fourth Thursdays of each month.

ARTICLE V. *Miscellaneous.*

Section 1. This club shall have the power to make rules for its meetings, levy assessments upon its members, and perform other acts not inconsistent with this constitution.

Section 2. One-third of all the members of the club shall constitute a quorum.

Section 3. Amendments to this constitution shall be offered in writing, shall lie upon the table for two weeks, and shall require for adoption a two-thirds vote of all members present.

Section 4. In all cases not otherwise provided for this club shall be governed by "Roberts's *Rules of Order*" as parliamentary guide.

During 1916–17 the following meetings (with an average attendance of 15 members) were held:

November 9, 1916: "How we have learned to count" by Dr. Nathan Altshiller, instructor in mathematics.

December 14: "Finger counting" by Harold Gimeno '18; "Finger calculation" by Enoch B. Ferrell '18.

January 16, 1917: "Mathematical wrinkles"[1] by Thomas L. Sorey '18 and Hugh S. Lieber '18.

February 13: "Some properties of triangles" by Professor Reaves.

March 13: "Paper folding" by Ella Mansfield '18.

March 27: "Life of Descartes" by Earl Bonham '17.

April 10: "Invention of logarithms" by Professor Edmund P. R. Duval; "If at the beginning of our era one cent had been placed at four per cent compound interest, what would be the radius of the gold sphere equivalent to the capital accumulated to date?" by Margaret Coleman '17.

April 17: "Mathematics and astronomy" by Professor Harry C. Gossard.

"Owing to the fact that most of the advanced students in mathematics, including both presidents of the club (Thomas L. Sorey '18 and Enoch B. Ferrell '18), entered military service, while the others left college, to fill the vacancies created by the military draft, it was decided to suspend the activities of the club for the year 1917–18."

TOPICS FOR CLUB PROGRAMS.

12. GEOMETRY OF FOUR DIMENSIONS.

By HENRY P. MANNING, Brown University.

If but a single meeting is to be devoted to this subject the most interesting way to take it up is to give a simple statement of theorems and proofs after the manner in which plane and solid geometry is studied in the schools. Most of those who belong to mathematics clubs have heard tales of strange things that are to be found in hyperspace, and what they would like to know is the geometry itself and the mysterious paths by which these things are found. Probably the best topics to take up would be perpendicularity and various kinds of angles, and then, if there is time, some of the theorems about parallel lines, planes and hyperplanes.[2]

[1] A book entitled *Mathematical wrinkles for teachers and private learners* was published by Samuel I. Jones at Gunter, Texas, in 1912.

[2] A list of theorems as I have taken them up on two or three occasions is given in *The Mathematics Teacher*, Vol. 7, Dec., 1914, pp. 49–58.

Such a treatment ought to be the central or main part of any study of this subject, but if there is time to look at it from other points of view the following topics (arranged under 7 headings) may be considered.[1]

1. The history of the idea of a fourth dimension. It will be convenient to think of three periods: down to 1827, from 1827 to 1870, from 1870 to the present day.

In the writings of Aristotle (384–322 B.C.), Pappus (end of the third century), and, according to Simplicius (sixth century A.D.), of Ptolemy (about 150 A.D.) there are speculations about the number of dimensions of space and the possibility or impossibility of there being more than three. The same matter is referred to by the English philosopher Henry Moore (1614–1687), and by Kant (1724–1804).[2]

In the development of algebra, which at first ascribed a geometrical meaning to all of its terms, there came about a reluctant extension of these terms beyond those which describe figures of two and three dimensions.[3]

Finally, there is a suggestion by d'Alembert (1717–1783), and after him by Lagrange (1736–1813), that time might be regarded as the fourth dimension.[4]

In the second period certain leading mathematicians took up particular parts of hypergeometry and proved theorems or developed formulæ.[5] These developments were mostly by analytical methods, but sometimes synthetic methods were used, and in them all the geometrical conceptions were made quite clear. There was also, however, much work done in pure analysis with equations and algebraic forms in n variables, where the language of geometry is sometimes used, but the conceptions of geometry are almost entirely absent.[6]

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[1] Most of the material for this work will be found in the author's *Geometry of Four Dimensions*, Macmillan, New York, 1914, and in a book of fourth dimension essays, *The Fourth Dimension Simply Explained*, Munn, New York, 1910. These will be referred to as *Geometry* and *Essays*.

[2] Pappus, after speaking of the three-line and four-line locus (see Heath's *Apollonius*, Cambridge, 1896, Introduction, chap. v) says that we cannot say of more than six lines, "the ratio of the content of four to that which is contained by the rest, since there is nothing contained by more than three distances. Men a little before our time have allowed themselves to interpret such things, signifying nothing at all comprehensible, speaking of the product of that which is contained by such lines into the square of this or the content of those. These things might, however, be stated and shown generally by means of compound ratios"—Hultsch ed., Vol. II, Berlin, 1877, p. 680. For other references see *Geometry*, pp. 1–3.

[3] *Geometry*, pp. 2–3.

[4] *Geometry*, p. 4; R. C. Archibald, "Time as a Fourth Dimension," *Bulletin of the American Mathematical Society*, Vol. 20, 1914, p. 409.

[5] Möbius, *Der barycentrische Calcul*, Leipzig, 1827, § 140, p. 184; Cayley, "Sur quelques théorèmes de la géométrie de position," *Crelle's Journal*, Vol. 31, pp. 213–226, in particular, pp. 217–218; Sylvester, "On the Centre of Gravity of a Truncated Triangular Pyramid," *Philosophical Magazine*, fourth series, Vol. 26, Sept., 1863, pp. 167–183 or *Mathematical Papers*, Vol. II, No. 65; Clifford, *Educational Times*, Jan., 1866, or *Mathematical Reprints*, Vol. 6, pp. 83–87, or *Mathematical Papers*, London, 1882, p. 601.

[6] Green, *Mathematical Papers*, edited by N. M. Ferrers, London, 1871, p. 188; C. G. J. Jacobi, "De binis quibuslibet functionibus homogeneis," *Crelle's Journal*, Vol. 12, 1834, p. 1; Cayley, two papers published in the *Cambridge Mathematical Journal*, Vol. 3, 1841, or *Mathematical Papers*, Vol. I, Nos. 2 and 3; Schläfli, (1) "Ueber das Minimum des Integrals," etc., *Crelle's Journal*, Vol. 43, 1852, pp. 23–36; (2) "On the Integral," etc., *Quarterly Journal*, Vols. 2 and 3, 1858–1860.

For further details see *Geometry*, pp. 4–8.

Since 1870 we have witnessed a very rapid growth of the subject. References are given in the *Geometry* to Veronese, Halphen, Jordan, Stringham, Hathaway, Cole (p. 142), Keyser, Craig, and others. Nearly every meeting of a mathematical society has some paper, and nearly every volume of a mathematical journal contains some article bearing on this subject.

2. The many curious phenomena in space of four dimensions make a description of this subject especially interesting, even to those who do not care for mathematics. An account of these phenomena after the manner of the numerous "non-mathematical" articles that have been written would furnish an entertaining paper for a mathematics club. It is easy to write such a paper because almost everything in this geometry is analogous to things of plane and solid geometry. In fact, the mathematician in developing and establishing its theorems is all the time consciously or unconsciously using these analogies. Even the proofs are often analogous to the proofs of the lower geometries, although the analogy itself is never a proof.[1]

3. The analogy with two-dimensional beings has been used very freely to show our relation to four-dimensional space.[2] In particular, this idea has been developed in the form of a romance by E. A. Abbott, in *Flatland*, and by C. H. Hinton in *An Episode of Flatland*. A report on these books is sometimes made the subject in a meeting of a mathematics club. The former has also been dramatized.[3]

The question whether anyone is able, or can acquire the ability, to form a conception of space of four dimensions may depend partly on what we mean by the word conception. It is certainly possible by practice to acquire a facility in describing figures of such a space and in stating and proving the theorems of this geometry.[4]

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[1] The following are some of the phenomena referred to. They are explained many times in the *Essays* and in all articles of a similar nature. The new perpendicular direction, *Essays*, pp. 45, 71, 82, 122. Passing out of a closed room, *Essays*, pp. 116, 123, 138, 149, 180, 188. Making symmetrical figures coincide by turning one of them over through hyperspace, *Essays*, pp. 48, 76, 95, 139, 158, 213. Turning a flexible hollow shell inside out, *Essays*, pp. 49, 170, 249. Knots, links of a chain, *Essays*, pp. 50, 145. Planes meeting only in a point, going around a plane, *Essays*, pp. 23, 24, 49. The hypercube, *Essays*, pp. 46, 72, 88, 92, 113, 147. Cutting apart a polyhedroid and spreading it out in a hyperplane, *Essays*, p. 74, *Geometry*, pp. 68, 238, 240, 248. Some of these subjects are mentioned by Professor Newcomb, "Philosophy of Hyperspace" (presidential address), *Bulletin of the American Mathematical Society*, Vol. 4, Feb., 1898, pp. 187–195; "Fairy land of Geometry," *Harper's Magazine*, Vol. 104, Jan., 1902, pp. 249–252, reprinted in *Sidelights on Astronomy*, New York, Harper and Brothers, 1906, pp. 155–164.

There is a story by Charles Johnstone, "Jones and the Fourth Dimension," *Harper's Magazine*, Vol. 129, June, 1914, pp. 117–122, that is rather amusing, though not quite accurate in its representation of a lack of distance in space of four dimensions.

[2] Essays, pp. 78, 81, 96, 115, 127, 155.

[3] Abbott, *Flatland*, London, 1884, Little, Brown and Company, Boston, 1907. Hinton, *An Episode of Flatland*, London, Swan, Sonnenschein and Co., 1907. In the *Mathematical Gazette*, Vol. 7, Jan., 1914, pp. 228–231, is an account of a dramatic performance of *Flatland* by a girls' school in Acton, England. This is reprinted in *School Science and Mathematics*, Vol. 14, Oct., 1914, pp. 583–587.

[4] See discussion of this question by C. J. Keyser, "Mathematical Emancipations," *Monist*, Vol. 16, 1906, pp. 65–83, particularly, pp. 81–82; reprinted in *The Human Worth of Rigorous Thinking*, New York, Columbia University Press, 1916, pp. 101–121. See also *Geometry*, pp. 15–16 and the last footnote on p. 9.

4. If the various topics suggested are taken up and several meetings are devoted to the geometry of four dimensions, the treatment of the geometry itself by synthetic methods as explained at the beginning of this article should come at this point.

5. The analytical geometry of four dimensions with formulæ for distances and angles of various kinds, the equations of hyperplanes, planes and lines, and a brief account of the hypersurfaces of the second degree, the hyperconicoids.[1]

6. Some particularly puzzling subjects may be suggested for further study.

Kinematics and mechanics of four dimensions promises to be very interesting. The cross section of a rope would be a sphere, not a circle. It would have to be fastened to a spherical ring, not to a circular ring. A chain could not be made of links, but alternate links and hollow spheres could be put together to form a chain. Instead of a two-dimensional wheel and a one-dimensional axle we must have a three-dimensional wheel, or a two-dimensional axis.[2]

If we lived on the three-dimensional boundary of a hypersphere, just as we actually live on the earth, held to it by an attraction towards the center, we could move in three directions. We might study the problem of changing pressure into motion (like that of the locomotive), or the problem of constructing a mill that will saw a hypersolid block into two pieces, just as we saw a log into boards.

Surfaces in hyperspace is another subject that needs to be studied a great deal. A curve in any kind of space extends only forwards and backwards and seems very much the same as in a plane as long as we confine our attention to the curve itself. It is only when we consider the space about the curve that we find in space of three dimensions the phenomena of knots and of torsion. So surfaces, considered by themselves, are very much the same in any space. As long as we confine our attention to the surface itself we find it two-dimensional and we move about on it very much as we do on a simple plane. There is no difficulty in understanding at least a small portion of an ordinary surface in any kind of space. But when we investigate the space about the surface, if the space is of more than three dimensions, we begin to be mystified. Rotation on an axis-plane and the

[1] This treatment is taken up by Jouffret, *Géométrie à quatre dimensions*, Paris, 1903, and (for the case of any number of dimensions) by Schoute, *Mehrdimensionale Geometrie, Sammlung Schubert*, XXXV and XXXVI, Leipzig, 1902 and 1905, I, § 6, pp. 125–180. Certain parts are worked out also in the following articles: F. N. Cole, "On Rotations in Space of Four Dimensions," *American Journal of Mathematics*, Vol. 12, 1890, pp. 191–210; T. Craig, "Displacements Depending on One, Two and Three Parameters in a Space of Four Dimensions," *American Journal of Mathematics*, Vol. 20, 1898, pp. 135–156; J. G. Hardy, "Curves of Triple Curvature," *American Journal of Mathematics*, Vol. 24, 1902, pp. 13–38; C. J. Keyser, "Concerning the Angles and the Angular Determination of Planes in 4-Space," *Bulletin of the American Mathematical Society*, Vol. 8, 1902, pp. 324–329. Two articles in the AMERICAN MATHEMATICAL MONTHLY may be mentioned here: B. H. Brown, "Centres of Similitude and their N-Dimensional Analogies," Vol. 23, May, 1916, pp. 155–159, and M. H. Sznyter, "Some Metrical Properties of the Pentahedroid in a Space of Four Dimensions," Vol. 24, Mar., 1917, pp. 113–119.

[2] See *Essays*, p. 31, Hinton, *The Fourth Dimension*, Swan, Sonnenschein and Co., London, 1904, pp. 31 and 71–73. The nature of rotations is taken up briefly at the end of an article by Beltrami, "Formules fundamentales de cinematique dans les espaces de courbure constante," *Bulletin des sciences mathématiques*, Vol. 11, 1876, pp. 233–240, *Opere*, Milan, Vol. III, 1911, pp. 23–29, and very fully by Cole, *loc. cit.*

fact that we can go around a surface just as in ordinary space we go around a line make one of its mysteries. The consideration of an entire surface that does not lie in any one three-dimensional space also involves peculiarities of hyperspace. An example is the surface of double revolution, a surface on which the ordinary plane geometry of Euclid holds true if we take for lines certain systems of circles. The study of surfaces is important in what is called analysis situs.[1]

7. The object and value of the study of this subject may be considered from three points of view: any direct applications that it may have, its relation to the science of geometry and to mathematics in general, and any philosophical discussions that it may involve.

We can use it to give simple proofs of certain theorems of ordinary geometry, such as the equivalence of symmetrical figures,[2] and theorems about certain figures that are sections or projections of simpler figures of hyperspace.[3] New figures have also been discovered in this way. Thus application has been made to architecture.[4] In mechanics it has been found useful to represent time as a fourth dimension. In this way the theory of relativity finds its simplest form of expression.[5] It can be used sometimes to give us graphical solutions of problems[6]. It helps us to visualize certain complicated groups of objects, such as determinants of more than three dimensions.[7] There are many interpretations of this geometry in which some other figure than the point is taken as element. These interpretations have been put forward to help us understand the geometry, but the relation can be reversed and the geometry applied to the study of these figures.[8] Its most important direct application lies in its furnishing names and concrete terms for certain complicated ideas and expressions of analysis.

In the next place we get a broader view of the nature of geometry by a study of geometry of four dimensions, and, as far as we can carry it, of higher dimensions. There are many theories in geometry that we hardly notice, if at all, in studying only plane and solid geometry. We understand better the foundations of geometry and we get an invaluable training in mathematical reasoning.[9]

Finally, the study of this geometry throws light on certain questions of philosophy, as, for example, the nature of space itself. The fact that we are able to develop other kinds of geometry besides the plane and solid geometry of our experience is put forward as disproving the theories of Kant on the subjective nature of our geometrical intuitions.

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[1] See *Geometry*, p. 219 and the reference there to Poincaré.

[2] *Geometry*, p. 149.

[3] See *Geometry*, p. 5, and the references there to Cayley and Veronese.

[4] Claude Bragdon, *Projective Ornament*, Manas Press, Rochester, N. Y., 1915.

[5] *Geometry*, p. 11.

[6] An example is Clifford's solution of a problem in probability referred to in *Geometry*, p. 5, and in footnote above.

[7] See abstract of paper by L. H. Rice, "Determinants of Many Dimensions," *Bulletin of the American Mathematical Society*, Vol. 23, 1916, p. 69.

[8] *Geometry*, p. 10.

[9] *Geometry*, p. 14.

COLLEGIATE MATHEMATICS FOR WAR SERVICE.

The chief aim of this department will be to gather and publish information concerning mathematical courses in preparation for war service. The editor invites the contribution of suitable articles and notes, and hopes, since timeliness will be important, that appropriate material will be sent in as soon as it becomes available. If an article is such that the Editors of the MONTHLY deem its early publication sufficiently pressing, preprints of it will be prepared. The editor of this department will also undertake to furnish to the extent of his means material and information desired by individual readers. Address Dr. Henry Blumberg, University of Illinois, Urbana, Ill.

COURSES IN COLLEGE IN PREPARATION FOR THE NAVY.[1]

BY R. G. D. RICHARDSON, Brown University.

The Need. To the colleges no call of the country in its war program is more urgent than that of the navy for commissioned officers. Besides the need of line officers for the new ships of the navy proper, the department has announced that it needs 22,000 commissioned officers to man the new mercantile marine which the navy will operate. Very few enlisted men have the education necessary to qualify for anything beyond the rank of petty officer. The Naval Academy can graduate at most a few hundreds yearly. The many auxiliary schools are filled beyond capacity but are turning out only a fraction of the number required. While a minimum of three to six months' sea experience has been a prerequisite, a large part of the theoretical training for candidates for the ensign's commission has in many instances been given in academic institutions. Since all candidates for commission as ensign must have the preliminary education required for admission to college, it is natural that the navy should look to us in increasing measure to supplement the efforts of the other agencies by giving courses which will aid in preparing men for this examination.

All college men who are physically fit must look forward to being called by their country to some arm of the service, and every college recognizes the imperative necessity of giving its students some technical preparation. Many men will prefer the navy to other branches of the service, and it devolves upon each school to ascertain how it can give the elements of the necessary training by adapting its present courses or adding new ones. This will in part anticipate any action which the government may deem it wise to take in directing the instruction in mathematics in those colleges whose students are furloughed for instruction.

What is Already Being Done. The experiences of other institutions furnish

[1] This brief sketch has been prepared very hastily and is subject to charges of inadequateness and inaccuracy. Its only excuse is that it may serve to call attention to the needs before definite programs for the academic year are made up. R. G. D. Richardson, Chicago, August 29 1918.

a basis for judgment as to what may be accomplished. Various types of work are already under way of which some examples may be cited.

Several institutions such as Harvard and Pennsylvania have loaned to the government their facilities for instruction and housing. Enlisted men who have the necessary basis of education and are deemed to possess the requisite personal qualifications are given training by officers of the navy to fit them for an ensign's commission or for the radio service.

In another group are institutions such as Yale, Princeton, Brown and the Universities of Michigan, California, and Washington which have regularly organized Naval Units. The formation of such Units is encouraged by the Navy Department but in most instances no official recognition has as yet been given. The units differ widely in the work they are attempting. While some students in a few engineering schools are being trained for the engineering branch, the greater number are headed for commissions in the line. In some cases a retired officer is assigned to service as head of the Unit and petty officers detailed to take charge of the drills. The greater part of the theoretical instruction, however, is given by members of the regular faculty. Candidates for commissions in the line take courses in Navigation, Seamanship, Navy Regulations, Ordnance, and Gunnery. Among the drills are day and night signalling of all kinds, handling of boats under oars and sail, School of the Company, Manual of Arms, School of the Section and Battery in Artillery, and Battleship Drills. Men who have enrolled in the Naval Reserve are given opportunity for practical work by means of week-end cruises or summer service on battleships or converted yachts.

Besides these two types of instruction, many institutions are adapting present courses or are introducing new ones with a view to giving preliminary training for future officers. Others are coöperating with outside agencies by furnishing mathematical instruction. The University of California supplements courses given to its undergraduates by courses in extension covering part of the same field as the naval units. During the present summer members of the Faculty of the University of Chicago and Northwestern University have been giving courses for those men, enlisted in the mercantile branch of the service, who have not yet been called to the school at Municipal Pier, Chicago. The Commandant of the station has detailed petty officers to supplement by drills the theoretical instruction in mathematics and astronomy.

Navigation. There seems to be no reason why a course in Navigation should not constitute a part of the mathematical program of every institution. Not only is the present demand very urgent but the new merchant marine will continue to call for officers in large numbers after the present war is won.

Prerequisite to such work is a course in Trigonometry with special emphasis on the derivation of those formulas used in computation and on the computation itself. The first topic to be discussed is the determination of position by Dead Reckoning including the methods of Plane, Mid-latitude and Mercator's Sailings. Double interpolation must be taught in connection with the Traverse Tables to be used in these computations. As a preliminary to

the study of the nautical astronomy involved, the student should possess a knowledge of the elements of spherical geometry and spherical trigonometry.[1] Of the latter the essentials include the formulas for right and quadrantal triangles and the Laws of Sines and Cosines together with the modifications of the latter used in calculating the haversine of a side or the haversine of an angle. [hav $x \equiv \frac{1}{2}$ versin $x \equiv \frac{1}{2}(1 - \cos x)$.] An introduction to some of the fundamental notions of astronomy is also necessary at this point.

Since from the standpoint of the student the most difficult ideas are those grouped about the notion of time considerable emphasis should be laid on this topic. One of the most important problems is the complete determination of the position at sea from altitude observations of celestial bodies. This is best accomplished by finding the intersection of two Sumner lines. A Sumner line or line of position is a small circle on the earth from any point of which the observed body will have the same altitude at the same instant. The center of the circle will be the point which has the observed body in its zenith. Perhaps the simplest application of nautical astronomy is the determination of latitude from a meridian altitude by means of the various corrections. For a meridian altitude the Sumner line will then be tangent to the parallel of latitude, and for a prime vertical observation, it will be tangent to the meridian. The most favored scheme for the reduction of the Sumner line observations is that of St. Hilaire. Out of such a computation comes as an almost immediate by-product the azimuth of the observed body. This gives at once the compass error which is another important problem of nautical astronomy.

The standard texts for this course will be cited in the specific instance below. Many books on navigation have been published within the past four years but most of them give rule of thumb methods rather than a mathematical treatment. A standard British text is by Gill (Longmans, Green & Co.) and from the standpoint of college instruction the last edition is very useful. Among other books to be recommended for the instructor are Martin's Navigation (Longmans, Green & Co.), Hall's Navigation (W. B. Clive, London), Muir's Navigation and Compass Deviations (U. S. Naval Institute, Annapolis, Md., $4.20).

The use of the compass, pelorus, vernier and chronometer and of the sextant with the real or an artificial horizon is necessary and to these may be added other instruments used at sea. It is essential that the student become familiar with the nature of charts of various kinds. Mercator Charts, Great Circle Charts and Position Plotting Sheets can be bought at cost from the Hydrographic Office (a catalog of all the publications of this office may be obtained for 30 c.).

If time is limited, fifty hours in Navigation with twice as much outside preparation on the part of the student will serve as a valuable introduction. In that time the central features can be covered. But it would be preferable to extend the course to ninety hours in order to take up the greater part of the topics covered by Bowditch, to acquire accuracy and facility in computation and

[1] Spherical Trigonometry is now a prerequisite for the naval school at Pelham Bay, L. I.

in handling the instruments, and to learn the use of the Kelvin-Aquino method for solving oblique-angled triangles by tables. The nature of the standard maintained by the Navy Department can be ascertained in this, as in other subjects, by an examination of the papers set for the ensign commission. Copies may be obtained on application to the Bureau of Navigation, Washington, D. C.

Other Courses. Since for the navy the target is generally in view, the problems of Ordnance and Gunnery are quite different from those of the army. From such books as Curtin's Naval Ordnance, Alger's Ground Work of Practical Naval Gunnery (U. S. Naval Institute, Annapolis, $4.85, $4.50) and the Bureau of Ordnance pamphlets the instructor could work up a few lectures on this topic. The elements consist largely in practical applications of arithmetic and trigonometry, and considerable practice in examples is necessary to insure correct understanding of the problems involved and to acquire facility.

The larger part of the training for officers is, of course, non-mathematical. While instructors competent to give extended courses in seamanship are not in general available and while many institutions will not find it practicable to obtain petty officers for the drills, some text-book work could in many cases be given to advantage. Instruction in the reading of charts may be supplemented by Buoy and Light Lists and the Coast Pilot. Charts on large and small scale obtainable from the Coast and Geodetic Survey are useful in laying out courses and in the study of the coast and harbors of some particular locality. Actual inspection of light-houses, buoys, and ships in process of construction would be a valuable auxiliary to instruction from the text. Elementary instruction in practical seamanship may be illustrated by means of models and by practise with small boats under oars and sail. Lectures on marine meteorology are valuable. Beside the Blue Jackets Manual (U. S. Naval Institute, 75 c.), which the men should know from cover to cover, students might be given parts of the U. S. Navy Regulations, Watch Officers Manual (U. S. Naval Institute, $1.10) and an elementary text in piloting, buoys, rules of the road, such as Chapman's Department Notes on Seamanship (U. S. Naval Institute, $1.00) contains valuable supplementary matter. Such work might extend to ninety hours. For more extensive instruction Knight's Seamanship is the standard text. Since company drill in the army is not essentially different from that in the navy (Landing Force Manual, U. S. Navy, 1918, $1.00) it may be substituted when no naval drill is possible. Swimming and naval calisthenics are possible substitutes for the regular gymnasium work.

A Specific Instance. As a specific instance of what is being done, the courses to be given to the Brown Naval Unit during 1918–19 may be cited. An elementary year course in Seamanship (Naval Science A, B, C) based on Chapman's Practical Motor Boat Handling (Motor Boating, New York City, $1.00) and Knight's Seamanship (Van Nostrand, $3.00) is required for the new men. For those who have had this elementary work, a year course (Naval Science D, E, F) is given including further topics in Seamanship, lectures and problems in Ordnance and Gunnery, and lectures by Rear Admiral Edwards, the Com-

mandant of the Unit, on engineering features. Men in this course will serve as officers in the drills. Material for the drills such as cutters, rifles, machine guns and artillery has been supplied by the government. The time devoted to class work together with the outside preparation and the drills (which are required of all) will in each of these courses exceed ten hours per week and three hours credit will be given toward the degree. All men are required to wear a uniform and if eighteen years of age will in all cases be enrolled in the Naval Reserve. Students thus enrolled will be taken for week-end cruises on naval vessels of various types.

A third course given for the Unit three hours throughout the year is Navigation A, B, C. This is open to those who have taken the freshman course in mathematics and by special permission to others who have had Plane Trigonometry. Bowditch's American Practical Navigator (U. S. Hydrographic Office, 1918, Washington, D. C., $2.25 +) is the text and the student must have the American Nautical Almanac (Nautical Almanac Office, Naval Observatory, Washington, D. C., 15 c. +) and the Altitude, Azimuth and Line of Position Tables (published by Hydrographic Office No. 200, 60 c. +). The first term's work consists of a study of the instruments involved, Dead Reckoning, Spherical Trigonometry and an introduction to Astronomy. In the second term determination of position at sea by observations on celestial bodies is studied, while in the third the remaining parts of Bowditch are covered and the Kelvin-Aquino method learned.

It is probable that for those men who can remain but one or two terms an abbreviated course in Navigation will be given. In that case only the main topics will be covered and the student will not have the opportunity to acquire the desired accuracy and facility.

All the distinctive text-books used at the Naval Academy and others of a similar character are in a special library open to the student.[1] Through the courtesy of the Yale authorities, men from the Brown Unit were invited to attend the Summer Nautical Training School which was held for the Yale Unit at Madison, Conn. In this school the students were under rigid naval discipline and received theoretical and practical instruction in Navigation, Seamanship, Naval Regulations, and Ordnance and Gunnery during the months of July and August.

[1] The names of the text-books used at Annapolis are given in the Annual Register U. S. Naval Academy (Government Printing Office, Washington, D. C.) a copy of which may be obtained on application to the Academy. For the Departments of Mathematics (beyond the calculus), Navigation, Seamanship, Ordnance and Gunnery the following are the texts: H. E. Smith's Analytic Mechanics, Alger's Hydromechanics, Smith's Strength of Materials, Bowditch's American Practical Navigator, White's Astronomy, Nautical Almanac, Azimuth Tables, Muir's Navigation and Compass Deviation, Logan's Marine Surveying, Practical Manual of the Compass, Stockton's Manual of International Law, Knight's Seamanship, The Deck and Boat Book, Signal Book U. S. Navy, Forms Procedure U. S. Navy, U. S. Naval Regulations, Grant's School of the Ship, Alger's Exterior Ballistics, Curtin and Johnson's Naval Ordnance 1915, Range and Ballistic Tables 1915, Gunnery Instructions 1913, Bureau of Ordnance Pamphlets, Official Publications.

The list of books published for camp libraries by the American Library Association Library War Service (Washington, D. C.) contains many titles in navigation and naval engineering, but it is not discriminating.

MATHEMATICAL INSTRUCTION AT THE GREAT LAKES NAVAL STATION.

By I. A. Barnett, Washington University, St. Louis.[1]

The work in mathematics at the Great Lakes Naval Station is at present under the auspices of the Y. M. C. A. as one of the phases of the Association's general educational program. Among the 40,000 enlisted men there were, at the time the writer left the station (August 15, 1918), about 800 studying mathematics. Their previous training ranged from a grammar-school to a college education. Attendance in the Y. M. C. A. classes was entirely voluntary. About half of those who attended were preparing themselves for the competitive entrance examinations of the Ensign School with a view, after four months' stay in this school, to an Ensign's commission. The subjects of examination are Geography, English, American history (with special emphasis on naval history), arithmetic, algebra (through quadratics), plane geometry and plane trigonometry. The remaining half consisted of men in the aviation school who wanted to learn some mathematics as an aid in their required studies, and to a smaller extent, of those who desired chiefly to advance their education.

There are now about 35 Y. M. C. A. classes in arithmetic, algebra and trigonometry. The topics discussed are as follows. Arithmetic (chiefly review, 6 hours including outside preparation): multiplication, division, g.c.d., l.c.m., fractions, decimals, mensuration, foreign money. Algebra (partly review, 15–20 hours): general nature, fundamental operations, linear equations and applications to problems, factoring, fractions, simultaneous linear equations, extraction of square roots. Trigonometry (mainly new, 10–12 hours): the right triangle and applications, logarithms, the general angle; no proofs. Outlines of these courses were mimeographed and distributed.

There are numerous difficulties in connection with the work. The teachers must be selected from the enlisted men, are usually inexperienced, and are not provided with extra time for the preparation of lessons. The classes are heterogeneous. No special rooms are provided for instruction, which is given in the barracks, the Y. M. C. A. huts or the regimental headquarters, the men often sitting on the floor. Usually there is no time for study except between 6:30 P.M. and 8:00 P.M., and almost all the work must be concentrated during these hours. The men work hard during the day, and only those especially ambitious and energetic attend the classes.

The most obvious and effective way of overcoming these difficulties would be to secure official recognition, by the naval authorities, of the Y. M. C. A. instruction. The writer has been recently informed that a start in this direction has already been made.

THE NAVAL UNIT AT THE UNIVERSITY OF CALIFORNIA.

For the proposed naval unit at the University of California,[2] the following outline of courses preparatory for naval service has been provisionally made.

[1] [Dr. Barnett was engaged for a period of 7 weeks in the organization and supervision of the work here described. Editors.]

[2] We are indebted to Prof. T. M. Putnam for the information about the war courses at the University of California.

(*a*) *For Deck Officer:* Plane and Spherical Trigonometry and Elementary Theory of Map Construction, Introduction to Plane and Solid Analytic Geometry, Navigation and Nautical Astronomy, Naval History, Oceanography and Marine Meteorology, Seamanship, Naval Regulations, and Physical Education. With the exception of Physical Education, each course is given 3 hours weekly, $\frac{1}{2}$ year. There are also drills and practical work at sea, arranged with the coöperation of naval officers. (*b*) *For Engineer Officer:* As set forth under Course of Instruction on pp. 277–281 of the Annual Register of the U. S. Naval Academy, Annapolis (1917–18), and in addition, Naval Architecture and Marine Engineering. Only such work of the Annapolis curriculum can not be provided for as requires the actual use of Navy equipment not available to the University. This work might be arranged, however, in coöperation with the Commandant of the Twelfth Naval District.

The course in Navigation and Nautical Astronomy, which has been planned to conform as closely as possible to the standards, methods, and general procedure employed by the U. S. Navy, includes the following topics: time and the chronometer; the sextant and its adjustments; the compass (variation, deviation, Napier's diagram, various methods of swinging a ship for deviation, etc.); piloting; the sailings; latitude by meridian altitude, by Polaris, and by circum-meridian altitudes; longitude and time sights; azimuth and amplitude by tables and by computation; the New Navigation (Sumner line, method of St. Hilaire); full day's work at sea. Bowditch's American Practical Navigator, and additionally, the American Nautical Almanac, azimuth tables, tide tables, charts, etc., are used. Sextants, a liquid compass, and other apparatus are available for the practical aspects of the work.

In the Seamanship course, Knight's Seamanship, the Blue Jackets Manual and the Seamanship Department Notes of the U. S. Naval Academy are used. In the course on Naval History, the text-books formerly used by the U. S. Naval Academy were: (1) Short History of the U. S. Navy, by Clark, Stevens, Alden and Krafft; (2) Famous Sea Fights from Salamis to Tsushima by John R. Hale and (3) World Politics at the End of the Nineteenth Century by Paul S. Reinsch. Next term, Naval Powers in the Present War by Gill, Charles and Daran (1917) will be substituted for (3). Besides, the instructors frequently refer to such authors as Mahan, Corbett, etc.

To assist the U. S. Naval Reserve Force in preparing men to pass the examination for the ensign's commission, The University of California has been conducting, since September, 1917, extension courses in (*a*) Navigation and Nautical Astronomy, (*b*) Seamanship and Ordnance and (*c*) Naval Regulations. The equivalent of a high-school education has been a prerequisite; course (*a*) presupposes Trigonometry. A rough approximation of the total enrollment is 800.

NOTES.

At the request of the Commanding Officer of the United States Naval Auxiliary Reserve School at the Chicago Municipal Pier, the Department of Mathematics of the University of Chicago has during the summer conducted

four consecutive courses in Navigation for enlisted men awaiting their call to the Pier School, which coöperates with the Ensign School at Pelham Bay Park, N. Y. For enlistment at the Pier, men were required to be between the ages of 21 and 31 and expected to have had at least a brief course in Trigonometry. The course in Navigation, given three hours daily for three weeks, covered Dead Reckoning and selected topics of Nautical Astronomy (such as the determination of latitude, longitude, altitude, azimuth and time). Bowditch's American Practical Navigator was used as a basis, and numerous problems taken from a specially prepared list[1] were assigned for outside work. After July 9, a course of one hour on signalling was added, instruction being given by a man detailed by the Commanding Officer of the Pier School. The total enrollment for the four courses was approximately 550.

Similar courses of four weeks each were given by members of the Department of Mathematics of Northwestern University to a total of approximately 600 men; but on account of the hurried preparation of insertions in this department, further details could not be accurately obtained for the present issue.

The following extract from a letter sent to the Editor of the MONTHLY by an esteemed correspondent contains suggestions that may prove helpful to those seeking college instructors in mathematics at the present time: "(1) There is practically no visible supply of young instructors of ordinary qualifications for instructorships. (2) Other university teachers may be willing to teach at least trigonometry; but care should be exercised to see that (*a*) the offer of help is really final and not to be presently withdrawn (I have suffered); (*b*) that the same teacher can proceed with higher courses as we reach them; . . . (*c*) that the proposed teacher will not himself be overloaded with classes and hence be unable to help in mathematics. (3) A source of supply of teachers that may be overlooked lies in those high-school teachers of university training who would ordinarily (*a*) not accept on account of low salaries; (*b*) not be acceptable as university teachers. This situation is now quite upset, and teachers may be available because (*a*) ordinary requirements cannot be insisted upon by university authorities (*e.g.*, Ph.D. degree *must* be waived); . . . (*b*) there is a strong patriotic motive for the high-school teacher to accept lower salary, since S. A. T. C. work is essentially *war* service; (*c*) school boards and superintendents will release men more readily for what is essentially *war* work."

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Ind.

Dr. F. S. NOWLAN, of Columbia University, has been appointed instructor in mathematics at Bowdoin College.

Professor T. O. WALTON, of William and Vashti College, has been elected professor of mathematics at the Colorado School of Mines.

[1] Copies of this list can be obtained by applying to Prof. E. H. Moore, University of Chicago.

Associate Professor D. D. LEIB has been promoted to a professorship of mathematics at Connecticut College.

Professor H. R. PHALEN, of Berea College, has been appointed instructor in mathematics at Armour Institute of Technology.

Mr. G. H. SCOTT has resigned the principalship of Benzonia Academy to accept the professorship of mathematics in Doane College.

Dr. G. W. SMITH, instructor at Beloit College, has been appointed instructor in mathematics at the University of Kentucky.

Assistant Professor LOUIS LINDSEY has been promoted to an associate professorship of applied mathematics at Syracuse University.

Professor F. E. CHAPMAN has resigned his position at Southern University to become head of the department of mathematics in the Gulf Coast Military and Naval Academy at Gulfport, Mississippi.

Professor H. E. HAWKES, who has been acting dean of Columbia College, has been appointed dean of the college, consequent upon the resignation of Dean KEPPEL who has become third assistant secretary of war.

At Colorado College, Professor C. H. SISAM, of the University of Illinois, has been appointed professor of mathematics and head of the department, and Dr. W. V. LOVITT, of Purdue University, has been appointed assistant professor of mathematics.

Professor H. L. RIETZ, of the University of Illinois, has accepted the appointment as professor and head of the department of mathematics at the University of Iowa, succeeding the late Professor A. G. SMITH, whose death occurred in 1916.

Associate E. W. CHITTENDEN of the University of Illinois has been appointed assistant professor of mathematics at the University of Iowa.

Mr. J. W. LASLEY has been promoted from instructor to assistant professor of mathematics at the University of North Carolina, and granted leave of absence to study at the University of Chicago.

At the University of California, Associate Professor D. N. LEHMER has been promoted to a professorship of mathematics and Instructors B. A. BERNSTEIN, THOMAS BUCK and FRANK IRWIN have been promoted to assistant professorships. After forty-five years of service, Professor G. C. EDWARDS has been made professor emeritus.

At the University of Illinois the following promotions have been made in the department of mathematics: Associate Professor J. B. SHAW to become full professor; Assistant Professor R. D. CARMICHAEL to become associate professor; Associates G. E. WAHLIN and A. J. KEMPNER to become assistant professors; Assistant R. F. Borden to become instructor. At the same institution the

following new appointments have been made: Associate Professor A. B. COBLE of Johns Hopkins University to be professor of mathematics; Associate Professor H. BLUMBERG of Nebraska to be associate in mathematics; Instructor JOSEPH B. ROSENBACH of New Mexico to be assistant in mathematics.

Dr. HORACE LAMB, professor of mathematics at the University of Manchester, has been appointed Halley lecturer at Oxford University.

Professor R. D. CARMICHAEL has placed in the hands of the Secretary of the Association his resignation as editor-in-chief of the MONTHLY, the same to take effect immediately after the publication of the December, 1918, number. His resignation grows out of his desire to have a greater portion of his energies free for the pursuit of his researches. A committee consisting of H. E. SLAUGHT, R. D. CARMICHAEL and E. R. HEDRICK has been appointed to select his successor.

Dr. R. S. HEATH, professor of mathematics at Birmingham University, has retired from active service on account of ill health.

Mr. CHARLES E. FLANAGAN, actuary of the Conservative Life Insurance Company, Wheeling, W. Va., a charter member of the ASSOCIATION and a contributor to the MONTHLY, died on April 22, at the age of 63 years.

It is announced in *Science* that Professor MAXIME BÔCHER of Harvard University has died at the age of fifty-one years.

Professor A. L. DANIELS, Williams professor of mathematics at the University of Vermont, died on July 18, at the age of 69 years. After a service of twenty-nine years, Professor Daniels was retired in 1914 as professor emeritus on the Carnegie Foundation.

The following items have been received concerning teachers of mathematics who have entered the service of the government:

Professor W. E. MILNE, of Bowdoin College, has joined the ordnance reserve corps with the rank of first lieutenant.

Dr. PAUL R. RIDER, of the department of mathematics at Washington University, has entered the national service.

Mr. THEODORE DOLL, who has been a fellow in mathematics in Northwestern University, and a member of the Association, is now enlisted in the service and assigned to Camp Grant.

Dr. J. N. RICE, instructor in mathematics in the Catholic University of America, is now on leave of absence, serving in the National Army.

Mr. J. J. NASSAU, instructor in mathematics at Syracuse University, is a member of the 303rd regiment of engineers now in France.

Professor V. H. WELLS, of the University of Pittsburgh, has been commissioned a lieutenant in the science and research division of the signal corps.

Dr. T. M. SIMPSON, Jr., of the University of Texas, has been granted leave of absence for work with the Y. M. C. A. in France until the end of the war.

Miss LUCY T. DOUGHERTY, a charter member of the Association and teacher of mathematics in the Kansas City (Kansas) High School, has gone to France to engage in Red Cross work.

Dr. E. G. BILL, assistant professor of mathematics at Dartmouth College, has been granted leave of absence in order that he may render service to the Canadian government on the staff of the Military Service Council at Ottawa.

Associate Professor J. E. ROWE, of Pennsylvania State College, is doing mathematical research for the National Advisory Committee for Aeronautics and has been detailed to the Bureau of Standards.

At the University of Wisconsin special intensive courses in trigonometry, navigation and gunnery are to be offered during the coming year. A course in practical gunnery will be given by Assistant Professor H. C. WOLFF, who is spending the summer at Fortress Monroe in special preparation for this work. Professor C. S. SLICHTER will give a course in ballistics.

The concluding number of Volume 24 of the *Bulletin of the American Mathematical Society* contains the twenty-seventh annual list of papers, giving the author, title and place of publication of the 108 papers presented to the Society and published during the year 1917–18.

The name of Professor FELIX KLEIN, of the University of Göttingen, together with those of six other German educators, has been cancelled from the roll of honorary members of the National Education Association in response to a persistent demand from active members of the association, from members of the Council of National Defense, and from others.

The American Association for the Advancement of Science will hold its seventy-first meeting in Baltimore from December 27 to 31, 1918, under the auspices of the Johns Hopkins University. Dr. JOHN MERLE COULTER of the University of Chicago, will preside. The address of the retiring president will be given by Dr. THEODORE W. RICHARDS, Director of the Wolcott Gibbs Memorial Laboratory, Cambridge. This seventy-first meeting of the American Association, which was established in 1848, will be marked by the importance of its program and by the increased interest manifested in all branches of the natural and applied sciences. It will embrace a program devoted very largely to definite working problems related to the winning of the war. When the Association met last in Baltimore, ten years ago, the membership of the Association was less than 6,000. The membership of the Association at present numbers nearly 15,000 and the coming meeting will be one of the most important gatherings of scientific men hitherto held in this country or elsewhere. The officers in charge of Section A, mathematics and astronomy, are members of the MATHEMATICAL ASSOCIATION OF AMERICA. They are Professor G. D. BIRKHOFF, of Harvard University, vice-president, and Professor F. R. MOULTON, of the University of Chicago, secretary.

The following graduate students have been granted the Doctorate in mathematics at American Universities since June, 1917, presenting disertations as indicated:

UNIVERSITY OF CALIFORNIA: FRANK R. MORRIS, "Classification of involutory cubic space transformations;" MARY HELEN SZNYTER, "The hypersurface of the second degree in four-dimensional space;" JAMES S. TAYLOR, "A set of five postulates for Boolean algebras in terms of the operation 'exception.'"

CATHOLIC UNIVERSITY OF AMERICA: OTTO J. RAMLER, "Three-cusped hypocycloids fulfilling certain assigned conditions."

UNIVERSITY OF CHICAGO: ISREAL A. BARNETT, "Differential equations with a continuous infinitude of variables;" JACOB M. KINNEY, "The general theory of congruences without any preliminary integrations;" ERNEST P. LANE, "Conjugate systems with indeterminate axis of curves;" JAMES E. MCATEE, "Modular invariants of a quadratic form for a prime power modulus;" WILLIAM P. OTT, "The general problem of the type of the brachistochrone with variable end points;" LEVI S. SHIVELY, "A new basis for the metric theory of congruences;" WEBSTER G. SIMON, "On the solution of certain types of linear differential equations in infinitely many variables."

COLUMBIA UNIVERSITY: GLENN JAMES, "Some theorems on the summation of divergent series."

CORNELL UNIVERSITY: H. H. DALAKER, "On the otomorphic functions of the group (0, 3; 2, 4, 6)."

UNIVERSITY OF ILLINOIS: RAYMOND FRANKLIN BORDEN, "On the Laplace-Poisson mixed equation;" HOBART DICKINSON FRARY, "The Green's function for a plane contour;" MERLIN GRANT SMITH, "On the zeros of functions defined by homogeneous linear differential equations containing a parameter."

UNIVERSITY OF PENNSYLVANIA: GEORGE H. HALLETT, "Linear order in three dimensional Euclidean and double elliptic space;" HARRY M. SHOEMAKER, "A generalized equation for vibrating membranes."

SYRACUSE UNIVERSITY: Mrs. EDWARD DRAKE ROE, Jr., "Interfunctional expressibility problems of symmetric functions."

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FERMAT'S METHOD OF INFINITE DESCENT.

By W. H. BUSSEY, University of Minnesota.

The method of infinite descent and the method of mathematical induction, which may appropriately be called the method of infinite ascent, both came into use in the seventeenth century in that productive period made famous by Descartes, Cavalieri, Pascal, Wallis and Fermat. Mathematical induction was first brought into prominence by Pascal, although it was invented by Maurolycus[1] and used in his arithmetic which was published in 1575. It is a method of infinite ascent from a special case, for which the theorem to be proved is verified by means of the argument from n to $n+1$. The method of infinite descent, which was invented by Fermat, is not so widely known. Fermat's own account of it is to be found in a letter entitled "*Relation des nouvelles découvertes en la science des nombres*" which he wrote to Pierre de Carcavi in 1659.[2] In this letter he tells Carcavi that he has discovered a new method of demonstration and applied it successfully to the solution of a considerable number of problems in the theory of numbers. He calls it the method of infinite or unlimited descent (descente infinie ou indéfinie) and says that at first he applied it only to negative propositions such as these two:

"*No number of the form* $3k-1$ *can be of the form* x^2+3y^2."

"*There is no right triangle whose sides are integers whose area is equal to the square of an integer.*"

He then gives the following abstract of his proof of the latter: "The proof is made by ἀπαγωγήν εις ἀδυνατον[3] in this manner: If there is a right triangle

[1] W. H. Bussey, "The Origin of Mathematical Induction," this MONTHLY, vol. 24, pp. 199–207.

[2] *Oeuvres de Fermat*, Vol. 2, pp. 431–436.

[3] Greek for *reductio ad absurdum*.